CECS 528, Exam 1 Part a, Fall 2024, Dr. Ebert

Directions: show all work.

Problem

LO1. Solve the following problems.

(a) Use the Master Theorem to determine the growth of $T(n)$ if it satisfies the recurrence $T(n) = 4T(n/2) + n^2 \log^2 n$. Defend your answer. (10 pts)

$n^{\log 4} = n^2$ = power function part of $f(n)$.

$\therefore$ By case 4 of M.T., $T(n) = \Theta(n^2 \log^3 n)$

(b) Use the substitution method to prove that, if $T(n)$ satisfies

$$T(n) = 2T(n/2) + \log n,$$

then $T(n) = O(n)$. (15 pts)

Assume $T(k) < ck + d\log n$ for some const. $c > 0$ and all $k < n$.

$T(n) = 2T(n/2) + \log n \le 2\left(\frac{cn}{2} + d\log\left(\frac{n}{2}\right)\right) + \log n$

$= cn + 2d\log n - 2d + \log n \le cn + d\log n$

$\iff d\log n - 2d + \log n \le 0 \iff$

$2d - d\log n \ge \log n \iff d(2 - \log n) \ge \log n$

$\iff d \le \frac{\log n}{2 - \log n} = \frac{1}{\frac{2}{\log n} - 1} \to -1$ which is

true iff $d \le -2$ and n is sufficiently large.

LO2. Solve each of the following problems.

(a) Recall that the `Find-Statistic` algorithm makes use of the Partitioning algorithm and uses a pivot that is guaranteed to have at least

$$3(\lfloor \frac{1}{2} \lceil \frac{n}{5} \rceil \rfloor - 2) \geq 3(\frac{1}{2} \cdot \frac{n}{5} - 3) = \frac{3n}{10} - 9 \geq \frac{n}{4}.$$

members of array a both to its left and to its right. Rewrite each of the above inequalities but now assuming the algorithm uses groups of seven instead of groups of five. Explain your reasoning for each of the numerical changes that you make. Also, the last inequality only holds for sufficiently large n. Determine a threshold k for which the inequality holds for $n \geq k$. (12 pts)

$4\left(\lfloor \frac{1}{2} \lceil \frac{n}{7} \rceil \rfloor - 2\right) \geq 4\left(\frac{1}{2} \cdot \frac{n}{7} - 3\right) = \frac{2n}{7} - 12 \geq \frac{n}{4}$

$\Leftrightarrow \left(\frac{2n}{7} - \frac{n}{4}\right) \geq 12 \Leftrightarrow \frac{8n - 7n}{28} = \frac{n}{28} \geq 12$

$\Leftrightarrow n \geq (28)(12) = 280 + 56 = 336.$

$3 \to 4$ since now 4 members of each group are $\leq$ or $\geq$ pivot. $5 \to 7$ since groups of 7 are now used.

(b) Consider the following algorithm of Karatsuba called `multiply` for multiplying two n-bit binary numbers x and y. In what follows, we assume n is even. Let x_L and x_R be the leftmost $n/2$ and rightmost $n/2$ bits of x respectively. Define y_L and y_R similarly. Let P_1 be the result of calling `multiply` on inputs x_L and y_L, P_2 be the result of calling `multiply` on inputs x_R and y_R, and P_3 the result of calling `multiply` on inputs $x_L + x_R$ and $y_L + y_R$. Then return the value $P_1 \times 2^n + (P_3 - P_1 - P_2) \times 2^{n/2} + P_2$. Demonstrate Karatuba's algorithm using $x = 78$ and $y = 129$. Show all work and steps. Please do *not* use a calculator. Hint: $78 \times 129 = 10062$. (13 pts)

First level of the algorithm:

$x = \underbrace{0100}_{x_L}\underbrace{1110}_{x_R}$

$y = \underbrace{1000}_{y_L}\underbrace{0001}_{y_R}$

$P_1 = (x_L)(y_L) = 0100000$ $\quad P_2 = (x_R)(y_R) = 1110$

$P_3 = (x_L + x_R)(y_L + y_R) = (10010)(1001) = (162)_2$

$(P_1 \times 2^8) + (P_3 - P_1 - P_2) \times 2^4 + P_2 = 2^{13}$

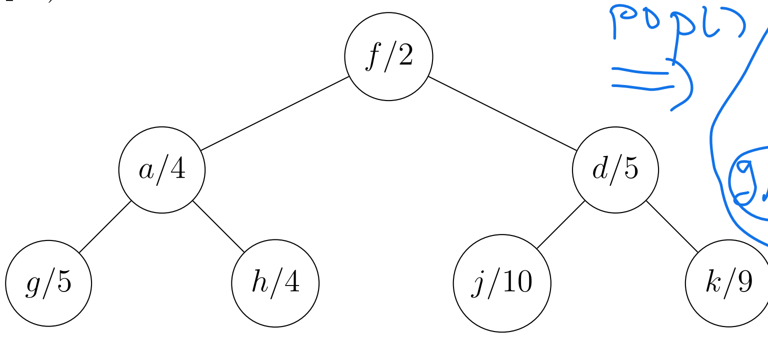
$+ (162 - 32 - 14) \times 16 + 14 = 10{,}062$

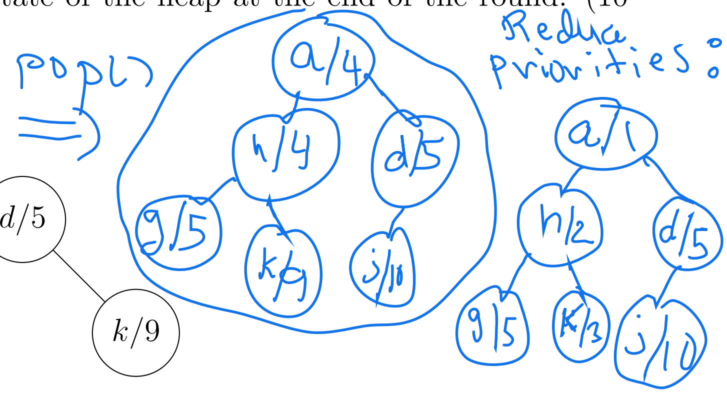
LO3. Solve each of the following problems.

(a) The tree T below shows the state of the binary min-heap at the beginning of some round of Prim's algorithm, applied to some weighted graph G. If G has undirected edges

$$(a, f, 1), (d, j, 5), (f, g, 7), (f, h, 2), (f, k, 3),$$

then draw (to the right of T) a plausible state of the heap at the end of the round. (10 pts)





(b) Recall that an instance of the `Task Selection` problem is a finite set T of tasks, where each task t has a start time $s(t)$ and finish time $f(t)$ that indicate the interval for which the task should be completed by a single processor. The goal is to find a subset T_{opt} of T of maximum size whose tasks are pairwise non-overlapping, meaning that no two tasks in T_{opt} share a common time in which both are being executed. Note: a task with respective start and finish times 2 and 4 does *not* overlap with a task with respective start and finish times 4 and 7, but *does* overlap with a task with respective start and finish times 3 and 6. For the algorithm described in the lecture notes, state the greedy choice that is being made in each step of the algorithm (5 pts) Chose the next task that starts after last finish time and ends earliest.

Apply the algorithm to the following set of tasks, where each triple in set T represents the id, start time, and finish time.

$$T = \{(1, 90, 120), (2, 110, 170), (3, 100, 120), (4, 20, 140), (5, 20, 70), (6, 40, 90), (7, 180, 190),$$

$$(8, 50, 170), (9, 60, 170), (10, 90, 200), (11, 20, 130), (13, 60, 150), (14, 30, 50), (15, 160, 170)\}.$$

Provide a table that, for each round of the algorithm, shows the value of any statistic that is used to make the greedy choice, and also the task that is selected for that round. (10 pts)

Last Finish	Selected
0	14
50	1
120	15
170	7
190	Finished

Optimal Set: {1, 7, 14, 15}