CECS 329, Exam 1, Fall 2025, Dr. Ebert

IMPORTANT: READ THE FOLLOWING DIRECTIONS SO YOU WILL NOT LOSE POINTS. Directions: This exam has SIX different problems: one problem for each of LO's 1-3 and three additional problems.

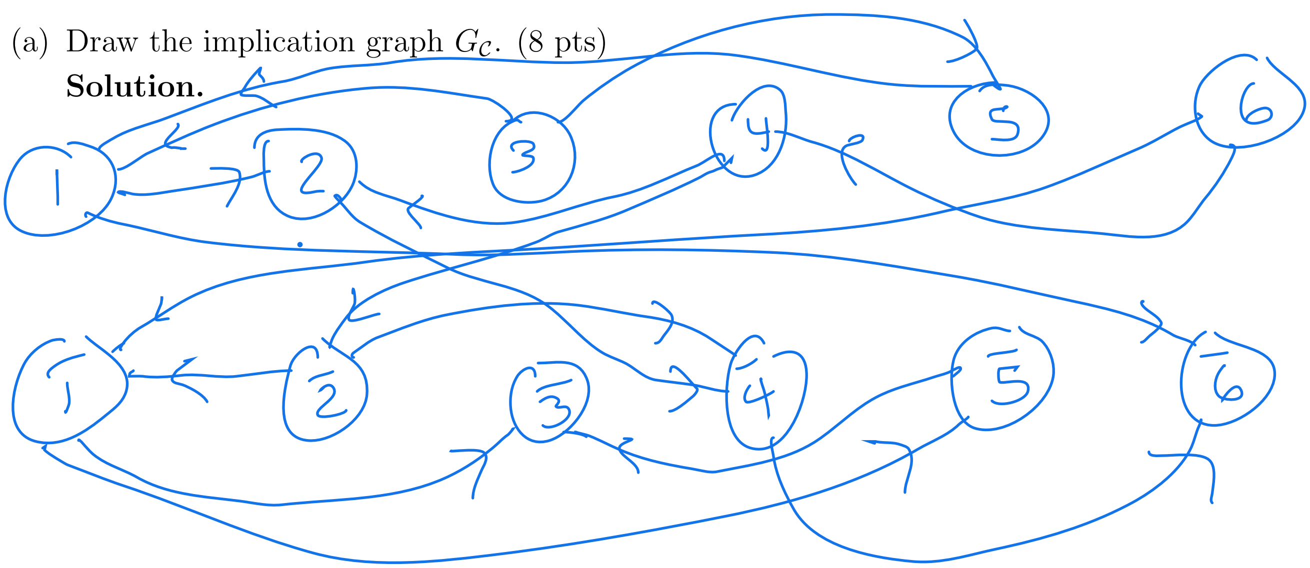
- For each problem, write your solution using **ONE SHEET OF PAPER ONLY (BOTH FRONT AND BACK)**. Write **NAME** and **PROBLEM NUMBER** on each sheet.
- Write solutions to different problems on **SEPARATE SHEETS** of paper.
- For example, if you decide to solve all six problems, then you will submit **SIX** sheets for grading.
- A 20% deduction in points will be applied to each solution that does not follow the above guidelines.

Unit 1 LO Problems (25 pts each)

LO1. Consider the `2SAT` instance

$$\mathcal{C} = \{(x_1, \overline{x}_3), (x_1, \overline{x}_5), (\overline{x}_1, x_2), (\overline{x}_1, \overline{x}_6), (x_2, \overline{x}_4), (\overline{x}_2, \overline{x}_4), (\overline{x}_3, x_5), (x_4, \overline{x}_6)\}.$$

(a) Draw the implication graph $G_{\mathcal{C}}$. (8 pts)
Solution.



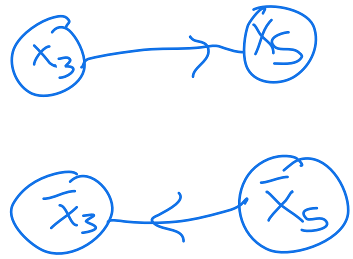
(b) Perform the `Improved 2SAT` algorithm by computing the necessary reachability sets. Use numerical order (in terms of the variable index) and positive literal before negative literal when choosing the reachability set to compute next. Draw the resulting reduced `2SAT` instance whenever a consistent reachability set is computed. Either provide a final satisfying assignment for $\mathcal{C}$ or indicate why $\mathcal{C}$ is unsatisfiable. (12 pts)

Solution.

$R_{x_1} = \{x_1, x_2, \overline{x}_4, \overline{x}_6\}$ is consistent.

$\alpha_{R_{x_1}} = (x_1 = 1, x_2 = 1, x_4 = 0, x_6 = 0)$

Reduced graph:



Satisfying assignment:

$\sigma = (x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 0, x_5 = 1, x_6 = 0)$

$R_{x_3} = \{x_3, x_5\} \Longrightarrow \alpha_{R_{x_3}} = (x_3 = 1, x_5 = 1)$

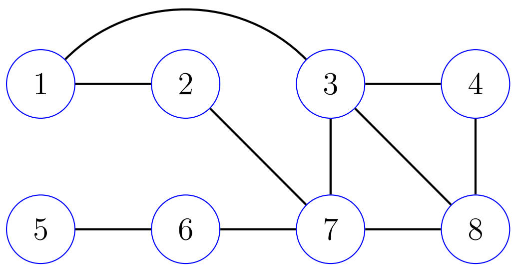
(c) Suppose instance $\mathcal{C}$ of `2SAT` is unsatisfiable and has 378 variables and 2892 clauses. What is the least possible number of queries to the `Reachability`-oracle that you would need make in order to confirm that $\mathcal{C}$ is unsatisfiable? Explain. (5 pts)

Solution. The least possible is 2 queries, for it may be the case that there is an inconsistent cycle that contains both x_1 and $\overline{x}_1$.

LO2. Do the following.

(a) Provide the definition of what it means to be a mapping reduction from problem A to problem B. Hint: do *not* assume that A and B are decision problems. (6 pts)

(b) The simple graph $G = (V, E)$ shown below is an instance of the `Hamilton Path (HP)` decision problem. Provide $f(G)$, where f is the mapping reduction from `HP` to `LPath` provided in lecture. (7 pts)



Solution. $f(G) = (G, k = 7)$.

(c) Verify that f is valid for input G from part b in the sense that both G and $f(G)$ have the same (decision) solution. Justify your answer. (12 pts)

Solution. G is a positive instance of `HP` since $P = 5, 6, 7, 8, 4, 3, 1, 2$ is an HP for G. Also, $f(G) = (G, k = 7)$ is a positive instance of `LPath` since P is a length-7 simple path.

LO3. An instance of `Set Cover` is a triple $(\mathcal{S}, m, k)$, where $\mathcal{S} = \{S_1, \ldots, S_n\}$ is a collection of n subsets, where $S_i \subseteq \{1, \ldots, m\}$, for each $i = 1, \ldots, n$, and k is a nonnegative integer. The problem is to decide if there are k subsets $S_{i_1}, \ldots, S_{i_k}$ from $\mathcal{S}$ for which

$$S_{i_1} \cup \cdots \cup S_{i_k} = \{1, \ldots, m\}.$$

In words, are there k members of $\mathcal{S}$ whose union covers the entire range of numbers from 1 to m? Thus, a certificate for instance $(\mathcal{S}, m, k)$ would be a collection $\mathcal{C} \subseteq \mathcal{S}$ of k subsets from $\mathcal{S}$. The following code takes as input instance $(\mathcal{S}, m, k)$ and certificate $\mathcal{C}$, and checks whether or not $\mathcal{C}$ is valid.

If $|\mathcal{C}| \neq k$, then Return 0.

Initialize array `found` so that `found`$[i] = 0$, for all $i = 1, 2, \ldots, m$.

For each $C \in \mathcal{C}$,

For each $j \in C$,

· `found`$[j] = 1$.

For each $i \in \{1, 2, \ldots, m\}$,

If `found`$[i] = 0$, then Return 0. //Number i was not covered by $\mathcal{C}$.

Return 1. //Each number in $\{1, 2, \ldots, m\}$ is covered by $\mathcal{C}$.

(a) Provide size parameters for the `Set Cover` problem and describe what each represents in relation to a `Set Cover` problem instance. Hint: there are two of them. (6 pts)

Solution. n denotes the number of sets in $\mathcal{S}$, while m denotes the maximum size of any set in $\mathcal{S}$.

(b) Use the size parameters to provide the big-O number of steps that is required by the verifier to check the validity of a certificate. Justify your answer. (7 pts)

Solution. For the nested loop, the outer loop iterates $k = \mathrm{O}(n)$ times and, for each of its iterations, the inner loop iterates at most m times. The final loop iterates at most m times. Moreover, the operations performed in either loop involve array accesses which require $\mathrm{O}(1)$ steps. Therefore, the total number of steps is $\mathrm{O}(mn)$ which is a quadratic polynomial. Therefore, `Set Cover` is an `NP` problem.

(c) Classify each of the following problems as being in `P`, `NP`, or co-`NP` (3 points each).

i. An instance of the `Feedback Arc Set` decision problem is a pair (G, k), where $G = (V, E)$ is a directed graph, $k \geq 0$ is an integer, and the problem is to decide if G has k vertices such that when the vertices are removed from G (along with all the edges that are incident with those vertices), then G becomes acyclic (i.e. has no cycles).

ii. An instance of `Set TriPartition` is a set S of natural nummbers and the problem is to decide if the members of S can be partitioned into three disjoint sets A, B, and C, such that

$$\sum_{a \in A} a = \sum_{b \in B} b = \sum_{c \in C} c.$$

iii. An instance of `Connected` is a simple graph $G = (V, E)$ and the problem is to decide if G is connected, meaning that, for every pair of vertices u and v, there is some path that starts at u and ends at v.

iv. An instance of `Complementary` is a Boolean formula $F(x_1, \ldots, x_n)$ and the problem is to decide if, for every assignment α to the variables $x_1, \ldots, x_n$,

$$F(\alpha) \oplus F(\overline{\alpha})$$

evaluates to 1, where $\overline{\alpha}$ is obtained from α by flipping all the assignment bits. For example, if $\alpha = (x_1 = 1, x_2 = 0, x_3 = 1)$, then $\overline{\alpha} = (x_1 = 0, x_2 = 1, x_3 = 0)$.

Solution. `NP`, `NP`, `P`, co-`NP`

Additional Problems

A1. Answer the following.

(a) Suppose there is an algorithm that solves decision problem A. Why is it the case that $A \leq_{\mathrm{T}} B$ for any problem B? (5 pts)

Solution. By definition $A \leq_{\mathrm{T}} B$ iff there is an algorithm for solving instances of A and that makes *zero or more* queries to a B-oracle. Furthermore, since there is an algorithm for solving A, and this algorithm makes zero queries to a B-oracle, by definition it is true that $A \leq_{\mathrm{T}} B$.

(b) Suppose that for some unsatisfiable `2SAT` instance $\mathcal{C}$, its implication graph $G_{\mathcal{C}}$ has the inconsistent cycle $x_2, \overline{x}_3, x_5, \overline{x}_2, x_1, x_2$. List the clauses of $\mathcal{C}$ that a) are responsible for R_{x_2} being an inconsistent reachability set, and b) are responsible for $R_{\overline{x}_2}$ being an inconsistent reachability set. (10 pts)

Solution. i) The edges of the path $x_2, \overline{x}_3, x_5, \overline{x}_2$ are obtained from the clauses $(\overline{x}_2 \vee \overline{x}_3)$, $(x_3 \vee x_5)$, and $(\overline{x}_5 \vee \overline{x}_2)$. Notice that assigning $x_2 = 1$ forces $x_3 = 0$, which in turn forces $x_5 = 1$, which finally forces $x_2 = 0$, a contradiction. ii) Similarly, the edges of the path $\overline{x}_2, x_1, x_2$ are obtained from $(x_2 \vee x_1)$ and $(\overline{x}_1 \vee x_2)$ and both cannot be simultaneously satisfied when setting $x_2 = 0$.

(c) Suppose that reachability set R_l is consistent for some literal l and $x \in R_l$ for some variable x. Explain why assignment α_{R_l} satisfies the clause $(\overline{x} \vee y)$ where y is another variable and $y \neq x$. (10 pts)

Solution. Clause $(\overline{x} \vee y)$ yields the implication graph edge (x, y) and, since x is reachable from l, then so is y, meaning that $\alpha_{R_l}(y) = 1$. Therefore, α_{R_l} satisfies $(\overline{x} \vee y)$.

A2. Answer the following.

(a) Recall the mapping reduction $f : \texttt{VC} \to \texttt{HVC}$ from `Vertex Cover` to `Half Vertex Cover`. If G is a graph having 30 vertices and $k = 10$, describe the graph $f(G, k)$ in relation to G. Show all work. (10 pts)

Solution. Since $10 < 30/2 = 15$, $f(G, k)$ is the same graph as G but with J isolated triangles added to G, where J satisfies the equation

$$k + 2J = \frac{1}{2}(30 + 3J).$$

Thus, $J = 10$ isolated triangles are added to G to obtain $f(G, k)$ and G will have 10-cover for a 30-vertex graph iff $f(G, k)$ has a 30-cover for a 60-vertex graph, i.e. iff $f(G, k)$ has a half cover.

(b) Let n be an integer. Is $f(n) = n^2 + n + 1$ a valid mapping reduction from `Even` to `Odd`? Explain. (10 pts)

Solution. $f(n)$ is not a valid mapping reduction since, e.g., $n = 1$ is a negative instance for `Even`, yet $f(1) = 3$ is a *positive* instance of `Odd`, and so, by definition, f is not a valid mapping reduction.

(c) Is $S = \{2, 5, 13, 18, 28, 36, 43\}$ a positive or negative instance of `Set Partition`? Explain. (5 pts)

Solution. Since the sum of the numbers in S is an odd number, there is no way to partition S into two subsets A and B whose respective members sum to the same number. For this would imply that the sum of all numbers in S would be even.

A3. Answer the following.

(a) What are the two size parameters for the `Set Partition (SP)` decision problem? Use these parameters to describe the big-O number of steps needed to perform the mapping reduction from from `SP` to `Subset Sum` described in lecture. Justify your answer. (15 pts)

Solution. $n = |S|$, and b is the maximum number of bits needed to represent any number in S. The most costly part of the reduction involves summing all the numbers of S. Adding each one to a running sum requires at most $O(b + i)$ steps for the ith added number, $i = 1, \ldots, n$. Hence, the worst case running time is

$$O(nb + 1 + 2 + \cdots + n) = O(nb + n^2),$$

and so it is a polynomial-step mapping reduction.

(b) Explain why, if some decision problem L belongs to $\mathtt{P}$, then so does $\overline{L}$. Use this and the fact that $\mathtt{P} \subseteq \mathtt{NP}$ to prove that any problem L that is in $\mathtt{P}$ must also be in $\mathtt{NP} \cap \text{co-}\mathtt{NP}$. Carefully explain each of your reasoning steps. (15 pts)

Solution. If some decision problem L belongs to $\mathtt{P}$, then there exists a polynomial-step algorithm for deciding instances of L. Moreover, negating every `return` statement of the algorithm yields a polynomial-step algorithm for deciding $\overline{L}$. Also, $\mathtt{P} \subseteq \mathtt{NP}$ means that both L and $\overline{L}$ are in $\mathtt{NP}$. Thus, $\overline{\overline{L}} = L$ is in co-$\mathtt{NP}$. Therefore, L is in the intersection of the two complexity classes.