CECS 329, Exam 1, Friday Fall 2025, Dr. Ebert

IMPORTANT: READ THE FOLLOWING DIRECTIONS SO YOU WILL NOT LOSE POINTS. Directions: This exam has SIX different problems: one problem for each of LO's 1-3 and three additional problems.

- For each problem, write your solution using **ONE SHEET OF PAPER ONLY (BOTH FRONT AND BACK)**. Write **NAME** and **PROBLEM NUMBER** on each sheet.
- Write solutions to different problems on **SEPARATE SHEETS** of paper.
- For example, if you decide to solve all six problems, then you will submit **SIX** sheets for grading.
- A 20% deduction in points will be applied to each solution that does not follow the above guidelines.

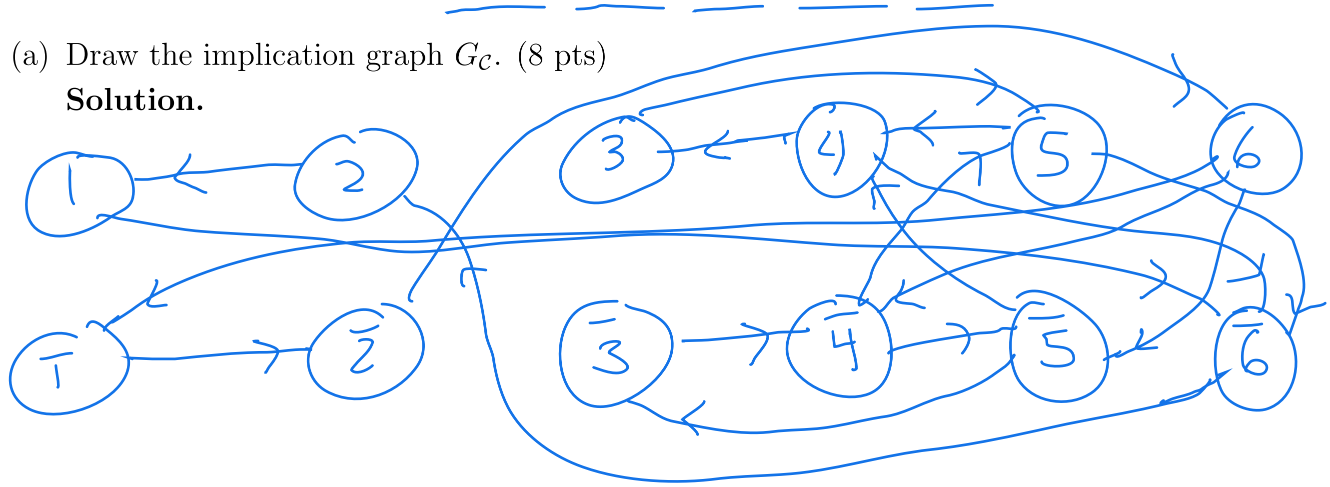
Unit 1 LO Problems (25 pts each)

LO1. Consider the 2SAT instance

$$\mathcal{C} = \{(x_1, \overline{x}_2), (\overline{x}_1, \overline{x}_6), (x_2, x_6), (x_3, \overline{x}_4,), (\overline{x}_3, x_5), (\overline{x}_3, \overline{x}_5), (x_4, x_5), (x_4, \overline{x}_5), (\overline{x}_4, \overline{x}_6), (\overline{x}_5, \overline{x}_6)\}.$$

(a) Draw the implication graph $G_{\mathcal{C}}$. (8 pts)

Solution.

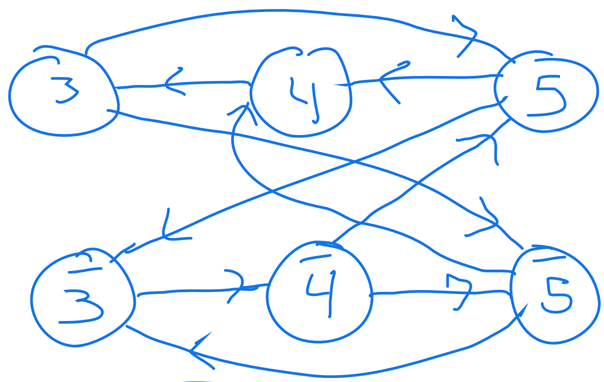


(b) Perform the `Improved 2SAT` algorithm by computing the necessary reachability sets. Use numerical order (in terms of the variable index) and positive literal before negative literal when choosing the reachability set to compute next. Draw the resulting reduced `2SAT` instance whenever a consistent reachability set is computed. Either provide a final satisfying assignment for $\mathcal{C}$ or indicate why $\mathcal{C}$ is unsatisfiable. (12 pts)

Solution.

$R_{x_1} = \{x_1, x_2, \bar{x}_6\}$ is consistent. $\alpha_{R_{x_1}} = \{x_1 = x_2 = 1, x_6 = 0\}$

Reduced graph:



$R_{x_3} = \{x_3, \bar{x}_5, \bar{x}_5, \bar{x}_3, x_4, \bar{x}_4\}$ is inconsistent.

$R_{\bar{x}_3} =$ ―― is also inconsistent. $\therefore \mathcal{C}$ is unsatisfiable

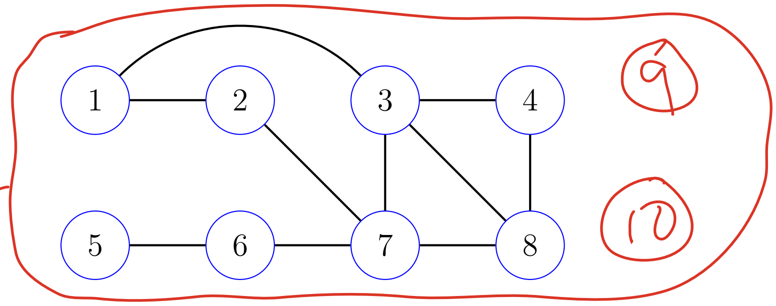
(c) Suppose instance $\mathcal{C}$ of `2SAT` is satisfiable and its implication graph $G_{\mathcal{C}}$ has path $x_2, \overline{x}_4, x_6, \overline{x}_2$. Given this information, what can you say about satisfying assignment α? Explain. (5 pts)

Solution. Since R_{x_2} is inconsistent ($\overline{x}_2$ is reachable from x_2 no assignment can set $x_2 = 1$ and satisfy all the clauses, including $(\overline{x}_6 \vee \overline{x}_2)$. Therefore, $\alpha(x_2) = 0$.

LO2. Do the following.

(a) Provide the definition of what it means to be a mapping reduction from problem A to problem B. Hint: do *not* assume that A and B are decision problems. (6 pts)

(b) Let $(G, k = 5)$ be an instance of `Vertex Cover (VC)`, where G is shown below. Provide $f(G, k)$, where f is the mapping reduction from `VC` to `Half Vertex Cover (HVC)` that was provided in lecture. (7 pts)

$f(G,k) =$ 

$\leftarrow G'$

Solution.

(c) Verify that f is valid for input G from part b in the sense that both G and $f(G)$ have the same (decision) solution. Justify your answer. (12 pts)

Solution. One can check that G does have a cover of size 4, namely $\{2, 3, 6, 8\}$. Thus, adding any additional vertex, say vertex 1, to this set yields a cover of size 5. Similarly, since no additional edges where added to $G' = f(G, k)$, it follows that $\{1, 2, 3, 6, 8\}$ is a half cover for G' since it is one half of 10 is the order of G'.

LO3. Given a simple graph $G = (V, E)$ a **cut** for G is a function $f : V \to \{0, 1\}$ that maps each vertex $v \in V$ to either 0 or 1. The **size** of a cut is equal to the number of edges $(u, v) \in E$ of G for which $f(u) \neq f(v)$. For example, for the graph appearing in the LO2 problem, suppose f maps each even-numbered vertex to 0, and every odd-numbered vertex to 1. To compute the cut size, we count how many edges there are between an even-numbered and odd-numbered vertex, and we see that the cut size equals 7. An instance of `Max Cut` is a pair (G, k) where G is a simple graph, and k is a nonnegative integer. The problem is to decide if G has a cut of size at least k.

Certificate. A certificate for instance (G, k) is a map $f : V \to \{0, 1\}$. Moreover, f is valid provided it yields a cut size of at least k.

Verifier. Below is the pseudocode that checks if f is a valid cut.

Initialize variable `sum` to 0.

For each edge $(u, v) \in E$

  If $f(u) \neq f(v)$, `sum`$++$.

Return (`sum` $\geq k$).

(a) Provide size parameters for the `Max Cut` problem and describe what each represents in relation to a `Max Cut` problem instance. (6 pts)

Solution. $m = |E|$, $n = |V|$.

(b) Use the size parameters to provide the big-O number of steps that is required by the verifier to check the validity of a certificate. Justify your answer. (7 pts)

Solution. The verifier iterates over each edge at most once. Moreover, checking $f(u)$, $f(v)$, comparing the values, and incrementing `sum` all require $O(1)$ steps, as does the checking the predicate that gets returned on the last line. Therefore, the verifier requires $O(m)$ steps which is linear. Therefore, the `Max Cut` problem is a member of complexity class `NP`.

(c) Classify each of the following problems as being in `P`, `NP`, or co-`NP` (3 points each).

i. An instance of the `Equal Numbers` decision problem is a pair of length-n integer arrays (a, b), the problem is to decide if a and b store the same numbers (but not necessarily in the same order).

ii. An instance of `Cubic Diophantine` is a triple (a, b, c, d) of natural numbers and the problem is to decide if there exist positive integers $x, y, z > 0$ for which

$$ax^3 + bx^2 + cx = d.$$

iii. An instance of `Set Cover` is a triple $(\mathcal{S}, m, k)$, where $\mathcal{S} = \{S_1, \ldots, S_n\}$ is a collection of n subsets, where $S_i \subseteq \{1, \ldots, m\}$, for each $i = 1, \ldots, n$, and k is a nonnegative integer. The problem is to decide if there are k subsets $S_{i_1}, \ldots, S_{i_k}$ from $\mathcal{S}$ for which

$$S_{i_1} \cup \cdots \cup S_{i_k} = \{1, \ldots, m\}.$$

In words, are there k members of $\mathcal{S}$ whose union covers the entire range of numbers from 1 to m?

iv. An instance of `Sum Avoidance` is a pair (S, k), where S is a set of natural numbers and k is some integer. The problem is to decide if no subset of S can sum to k.

Solution. `P`, `NP`, `NP`, co-`NP`

Additional Problems

A1. Answer the following.

(a) Provide the definition for what it means for problem A to be Turing reducible to problem B. (5 pts)

(b) If variable x occurs in some `2SAT` instance $\mathcal{C}$ and R_x is an inconsistent reachability set, explain why there is a path from x to $\overline{x}$. Hint: for some literal l, there must be a path P_1 from x to l, and a path P_2 from x to $\overline{l}$. (10 pts)

Solution. P_1 is a path from x to l. Also, the contrapositive of P_2 is a path from l to $\overline{x}$. Therefore, $P_1 \circ \overline{P}_2$ is a path from x to $\overline{x}$.

(c) Suppose that reachability set R_l is consistent for some literal l and $x \in R_l$ for some variable x. Explain why assignment α_{R_l} satisfies the clause $(\overline{x} \vee y)$ where y is another variable and $y \neq x$. (10 pts)

Solution. Clause $(\overline{x} \vee y)$ yields the implication graph edge (x, y) and, since x is reachable from l, then so is y, meaning that $\alpha_{R_l}(y) = 1$. Therefore, α_{R_l} satisfies $(\overline{x} \vee y)$.

A2. Answer the following.

(a) Recall the mapping reduction $f : \texttt{SS} \rightarrow \texttt{SP}$ from `Subset Sum` to `Set Partition`. If the numbers in set S sum to 365 and $t = 166$, then describe the set $f(S, t)$ in relation to S. Show all work. (10 pts)

Solution. $t = 166 < 365/2$ implies that $f(S, t) = S \cup \{J\}$, where $J = M - 2t = 365 - 332 = 33$.

(b) Let n be an integer. Is $f(n) = n^3 + n^2 + n + 1$ a valid mapping reduction from `Even` to `Odd`? Explain. (10 pts)

Solution. Since the cube and square of an even number n are both even, we have $f(n) = \text{even} + \text{even} + \text{even} + 1$ equals an odd number. Thus, a positive instance of `Even` maps to a positive instance of `Odd`. Similarly, since the cube and square of an odd number n are both odd, we have $f(n) = \text{odd} + \text{odd} + \text{odd} + 1$ equals an even number. Thus, a negative instance of `Even` maps to a negative instance of `Odd`.

(c) Is the graph that appears in the LO2 problem a positive instance of `Hamilton Path`? Defend your answer. (5 pts)

Solution. Yes, path $P = 2, 1, 3, 4, 8, 7, 6, 5$ is a simple path that visits every vertex of the graph.

A3. Answer the following.

(a) What are the two size parameters for the `Max Clique` decision problem? Use these parameters to describe the big-O number of steps needed to perform the mapping reduction from `Max Clique` to `Max Independent Set` described in lecture. Justify your answer. (15 pts)

Solution. Given instance $G = (V, E)$, $n = |V|$, $m = |E|$, the map reduction involves taking the complement $\overline{G}$ of G. Since the vertices of $\overline{G}$ are the same as for G, one can copy them in $\mathrm{O}(n)$ steps. Also, by making a table T in $\mathrm{O}(m)$ steps to store the edges of G, one can iterate through all unordered pairs (u, v), $u \neq v$, and add (u, v) to $\overline{G}$ iff $(u, v) \notin T$. This requires $\mathrm{O}(n^2)$ steps. Thus, the mapping reduction requires $\mathrm{O}(n+n^2) = \mathrm{O}(n^2)$ steps, and it is a polynomial-step reduction.

(b) Explain why, if some decision problem L belongs to $\texttt{P}$, then so does $\overline{L}$. Use this and the fact that $\texttt{P} \subseteq \texttt{NP}$ to prove that any problem L that is in $\texttt{P}$ must also be in $\texttt{NP} \cap \text{co-}\texttt{NP}$. Carefully explain each of your reasoning steps. (15 pts)

Solution. If some decision problem L belongs to $\texttt{P}$, then there exists a polynomial-step algorithm for deciding instances of L. Moreover, negating every `return` statement of the algorithm yields a polynomial-step algorithm for deciding $\overline{L}$. Also, $\texttt{P} \subseteq \texttt{NP}$ means that both L and $\overline{L}$ are in $\texttt{NP}$. Thus, $\overline{\overline{L}} = L$ is in co-$\texttt{NP}$. Therefore, L is in the intersection of the two complexity classes.