CECS 528, Homework Assignment 3, Fall 2025, Dr. Ebert

Directions: Please review the `Homework` section of the syllabus including a list of all rules and guidelines for writing and submitting solutions.

Due Date: Saturday, November 15th as a PDF-file upload to the HW3 Canvas dropbox.

Problem

Recall the `LPath` decision problem from the mapping reducibility lecture. An AI chatbot was asked the following question. How can I provide a polynomial-step mapping reduction from `LPath` to `Hamilton Path`? The chatbot responded with the following explanation.

To reduce the LPATH problem to the Hamilton Path problem, you can transform an instance $(G = (V, E), k)$ into a new graph G' such that G has a simple path of length k iff G' has a Hamiltonian path. **This is a valid reduction because the LPATH problem is NP-complete and the Hamiltonian Path problem is a specific case of the LPATH problem where the path length is** $(|V| - 1)$. Here is how to construct G'. Create a new graph $G' = (V', E')$ by adding a set of $(m = |V| - 1 - k)$ new vertices to G. Here,

$$V' = V \cup \{u_1, u_2, \ldots, u_m\},$$

where $u_i \notin V$ for all i. The original edges E are preserved in E'. Add edges such that every new vertex u_i is connected to every other vertex in V' (including all original vertices in V and all other new vertices). The set of new vertices forms a clique that is also fully connected to the original graph. The total number of vertices in G' is

$$n' = |V| + m = |V| + (|V| - 1 - k) = 2|V| - 1 - k.$$

A Hamiltonian path in G' must visit every one of the n' vertices exactly once. The auxiliary vertices form a "super-connected" component. Any path entering this component can traverse all the auxiliary vertices in any order due to the clique structure, and can enter/leave the original graph G through any vertex. Any simple path of length k in the original graph G can be extended to a Hamiltonian path in G' by appending a path that traverses all the m auxiliary vertices and any remaining original vertices not in the length k path, which is always possible because of the high connectivity of the new vertices. Conversely, if G' has a Hamiltonian path, that path must contain all vertices of V. The segment of this path that stays within the original vertices V forms a simple path of length at most $(|V| - 1)$. The construction ensures that the length condition in G corresponds directly to the Hamiltonian path condition in G'. The construction takes polynomial time because adding vertices and edges based on the size of the graph is efficient. The result is an instance of the Hamiltonian Path problem where G' has a Hamiltonian path if and only if the original graph G has a simple path of length k.

1. Explain why the sentence printed in boldface adds nothing useful to the explanation. Hint: there are two different issues that you should identify, and one of them is *not* because the reduction is actually invalid (see next problem). (10 pts)

2. Provide an instance of `LPath` for which the described mapping reduction is invalid, meaning that it either maps a positive instance to `LPath` to a negative instance of `HP` or maps a negative instance of `LPath` to a positive instance of `HP`. (10 pts)

3. Describe a correct mapping reduction from `LPath` to `HP`. Prove that your reduction is correct. Hint: although the chatbot's answer is incorrect, it provides a good starting point for designing a correct reduction. (20 pts)

4. Apply your reduction to the counterexample you provided in problem 3 and verify that the reduction is valid. (10 pts)