Review of Big-O Notation

Last Updated: August 8th, 2026

1 Big-O Notation

Big-O notation is useful for making succinct statements about the growth of a function $f(n)$, where n is a natural number. It is succinct because it uses the simplest representative from a general family of functions. For example, consider the family of quadratic functions. The general description of this family is that it represents the set of all functions of the form

$$f(n) = an^2 + bn + c,$$

where a, b, and c are constants. When using big-O notation we only care about the family of the function whose growth is being described. Indeed, we neither care about the constants, nor care about any lower order terms. For example, if we want to convey that a function $f(n)$ is bounded above by a quadratic function, then we would write

$$f(n) = \mathrm{O}(n^2).$$

This is because, if $f(n) = \mathrm{O}(n^2)$, then, by the definition of big-O (see below), $f(n)$ is big-O of *every* quadratic function.

Example 1.1. Carol has programmed the `Insertion Sort` algorithm to run on her laptop. What upper bound can she provide on the elapsed time $t(n)$ that will occur on her laptop clock after `Insertion Sort` has sorted an integer array of size n? She knows that the worst case occurs when the input array is sorted in reverse order, and in this case a total of

$$\frac{n(n-1)}{2} = \frac{n^2}{2} - \frac{n}{2}$$

comparisons and swaps must be performed in order to sort such an array. Thus the elapsed time on her clock will be roughly proportional to this amount. Therefore, she may write $t(n) = O(n^2)$. Notice that she does not include the constant of proportionality, because determining that value is highly dependent on a number of hidden factors, including her laptop's processor, other programs her laptop might be simultaneously running, the quality of the programming-language compiler, etc.. These details have nothing to do with the `Insertion Sort` *algorithm* being used. □

5 3 1 6 2

$t(n) = \Omega(n)$

lower bound

3 5 1 6 2

1 3 5 6 2

1 2 3 5 6

worst case: $n \quad n-1 \quad n-2 \quad \cdots \quad 2 \quad 1$

total swaps: $0 + 1 + 2 + \cdots + n-2 + n-1 =$

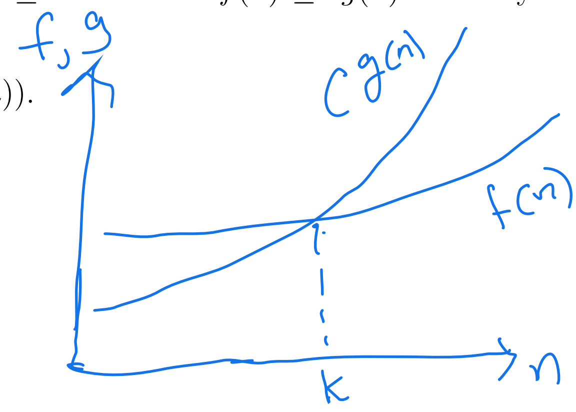
$$\frac{n(n-1)}{2} = O(n^2)$$

Let $f(n)$ and $g(n)$ be functions from the set of nonnegative integers to the set of nonnegative real numbers. Then

Big-O $f(n) = \mathrm{O}(g(n))$ iff there exist constants $c > 0$ and $k \geq 1$ such that $f(n) \leq cg(n)$ for every $n \geq k$.

Big-Ω $f(n) = \Omega(g(n))$ iff there exist constants $c > 0$ and $k \geq 1$ such that $f(n) \geq cg(n)$ for every $n \geq k$.

Big-Θ $f(n) = \Theta(g(n))$ iff $f(n) = \mathrm{O}(g(n))$ and $f(n) = \Omega(g(n))$.



little-o $f(n) = \mathrm{o}(g(n))$ iff $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

little-ω $f(n) = \omega(g(n))$ iff $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$.

$1 + o(n)$

2

Example 1.2. From the above discussion we have $f(n) = 3.5n^2 + 4n + 36 = \Theta(n^2)$ since

$$3.5n^2 \leq f(n) = 3.5n^2 + 4n + 36 \leq 3.5n^2 + 4n^2 + 36n^2 = 43.5n^2,$$

is true for all $n \geq 1$. And so $f(n) = \Theta(n^2)$, where $c_1 = 3.5$ and $c_2 = 43.5$ are the respective lower and upper-bound constants. $\square$

Definition 1.3. The following table shows the most common kinds of rules for $g(n)$ that are used within big-O notation.

Function	**Type of Growth**
1	constant growth
$\log n$	logarithmic growth
$\log^k n$, for some integer $k \geq 1$	polylogarithmic growth
n^k for some positve $k < 1$	sublinear growth
n	linear growth
$n \log n$	log-linear growth
$n \log^k n$, for some integer $k \geq 1$	polylog-linear growth
$n^j \log^k n$, for some integers $j, k \geq 1$	polylog-polynomial growth
n^2	quadratic growth
n^3	cubic growth
n^k for some integer $k \geq 1$	polynomial growth
$2^{\log^c n}$, for some $c > 1$	quasi-polynomial growth
$\omega(n^k)$, for all integers $k \geq 1$	superpolynomial growth
a^n for some $a > 1$	exponential growth

Example 1.4. Returning to Example 1.1, using big-O notation Carol can say that, when running `Insertion Sort` on her laptop with an input of size n, the elapsed time equals $\mathrm{O}(n^2)$ seconds. Also, since `Insertion Sort` requires at least n comparisons for any input, she may also say that its running time equals $\Omega(n)$ seconds. $\square$

Theorem 1.5. The following are all true statements.

1. $1 = o(\log n)$
2. $\log n = o(n^\epsilon)$ for any $\epsilon > 0$
3. $\log^k n = o(n^\epsilon)$ for any $k > 0$ and $\epsilon > 0$
4. $n^a = o(n^b)$ if $a < b$, and $n^a = \Theta(n^b)$ if $a = b$.
5. $n^k = o(2^{\log^c n})$, for all $k > 0$ and $c > 1$.
6. $2^{\log^c n} = o(a^n)$ for all $a, c > 1$.
7. For nonnegative functions $f(n)$ and $g(n)$,
$$(f + g)(n) = \Theta(\max(f, g)(n)).$$
8. If $f(n) = \Theta(h(n))$ and $g(n) = \Theta(k(n))$, then $(fg)(n) = \Theta((hk)(n))$.
9. If $f(n) = o(g(n))$ then $f(n) = \mathrm{O}(g(n))$.
10. If $f(n) = \omega(g(n))$ then $f(n) = \Omega(g(n))$.
11. If
$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = C > 0,$$
then $f(n) = \Theta(g(n))$.

Example 1.6. For each of the following, state whether $f(n) = \mathrm{O}(g(n))$, $f(n) = \Omega(g(n))$, or both, i.e. $f(n) = \Theta(g(n))$.

1. $f(n) = 3n + 5$, $g(n) = 10n + 6\log n$. $f(n) = \Theta(g(n)) = \Theta(n)$
2. $f(n) = \sqrt{n} \cdot \log^2 n$, $g(n) = \sqrt[3]{n}\log^3 n$. 2. $f(n) = \Omega(g(n))$.
3. $f(n) = 10\log n$, $g(n) = 50\log n^2$. 3. $f(n) = \Theta(g(n))$
4. $f(n) = n^2/\log n$, $g(n) = n\log^2 n$.
5. $f(n) = n2^n$, $g(n) = 3^n$.

4. $f(n) = \Omega(g(n))$

$$\frac{n^2}{\log n \; n \log^2 n} = \frac{n}{\log^3 n} \to \infty$$

$$\frac{n^{1/2}\log^2 n}{n^{1/3}\log^3 n} =$$

$$\frac{10\,\cancel{\log n}}{2 \cdot 50\,\cancel{\log n}} = \frac{10}{100}$$

$$\log n^a = a\log n$$

$$\frac{n^{1/6}}{\log n} \to \infty$$

$$5.\ \frac{n2^n}{3^n} = n\left(\frac{2}{3}\right)^n \to 0$$

5. $f(n) = O(g(n))$

$$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$$

2 Advanced Results

Log Ratio Test. Suppose f and g are continuous functions over the interval $[1, \infty)$, and

$$\lim_{n\to\infty} \log(\frac{f(n)}{g(n)}) = \lim_{n\to\infty} \log(f(n)) - \log(g(n)) = L.$$

Then

1. If $L = \infty$ then
$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty. \Rightarrow f(n) = \omega(g(n))$$

2. If $L = -\infty$ then
$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

3. If $L \in (-\infty, \infty)$ is a constant then
$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 2^L.$$

Integral Theorem. Let $f(x) > 0$ be an increasing or decreasing Riemann-integrable function over the interval $[1, \infty)$. Then

$$\sum_{i=1}^{n} f(i) = \Theta(\int_1^n f(x)dx),$$

if f is decreasing. Moreover, the same is true if f is increasing, provided $f(n) = O(\int_1^n f(x)dx)$.

Proof of Integral Theorem. We prove the case when f is decreasing. The case when f is increasing is left as an exercise. The quantity $\int_1^n f(x)dx$ represents the area under the curve of $f(x)$ from 1 to n. Moreover, for $i = 1, \ldots, n-1$, the rectangle R_i whose base is positioned from $x = i$ to $x = i+1$, and whose height is $f(i+1)$ lies under the graph. Therefore,

$$\sum_{i=1}^{n-1} \text{Area}(R_i) = \sum_{i=2}^{n} f(i) \leq \int_1^n f(x)dx.$$

$1+2+3+\cdots+n = \frac{n(n+1)}{2} = \Theta(n^2)$

Adding $f(1)$ to both sides of the last inequality gives

$$\sum_{i=1}^{n} f(i) \leq \int_1^n f(x)dx + f(1).$$

$\sum_{i=1}^{n} i = 1+2+\cdots+n$

Now, choosing $C > 0$ so that $f(1) = C\int_1^n f(x)dx$ gives

$\int_1^n x\,dx = \frac{x^2}{2}\Big|_1^n = \frac{n^2}{2} - \frac{1}{2} = \Theta(n^2)$

$$\sum_{i=1}^{n} f(i) \leq (1 + C)\int_1^n f(x)dx,$$

which proves $\sum_{i=1}^{n} f(i) = \mathrm{O}(\int_1^n f(x)dx)$.

Now, for $i = 1, \ldots, n-1$, consider the rectangle R'_i whose base is positioned from $x = i$ to $x = i+1$, and whose height is $f(i)$. This rectangle covers all the area under the graph of f from $x = i$ to $x = i + 1$. Therefore,

$$\sum_{i=1}^{n-1} \text{Area}(R'_i) = \sum_{i=1}^{n-1} f(i) \geq \int_1^n f(x)dx.$$

Now adding $f(n)$ to the left side of the last inequality gives

$$\sum_{i=1}^{n} f(i) \geq \int_1^n f(x)dx,$$

which proves $\sum_{i=1}^{n} f(i) = \Omega(\int_1^n f(x)dx)$.

Therefore,

$$\sum_{i=1}^{n} f(i) = \Theta(\int_1^n f(x)dx).$$

3 Big-O Notation within the Context of Algorithms

Given a problem L, and an algorithm $\mathcal{A}$ that solves L, big-O notation finds its use in the study of algorithms as a means for describing bounds on the number of steps and the amount of memory required by $\mathcal{A}$ as a function of the **size parameters** of L, i.e. the parameters used to indicate the number of bits required to represent an instance of L.

Example 3.1. The following are examples of how big-O notation arises in the study of data structures and algorithms.

1. Inserting an item into a balanced tree of size n requires $\mathrm{O}(\log n)$ insertions.
2. It has been proven that, sorting n numbers using pairwise comparisons requires $\Omega(n \log n)$ comparisions.
3. The Fast Fourier Transform algorithm has a running time of $\Theta(m \log m)$, where m is the degree of the input polynomial.
4. Two b-bit integers may be recursively added/subtracted in $\mathrm{O}(b)$ steps and recursively multiplied/divided in $\mathrm{O}(b^2)$.

It is also common to see big-O notation used within an arithmetic expression, such as $n^2 + o(n)$, and $n^{\mathrm{O}(1)}$.

In these examples, we see that big-O notation is useful for providing an upper bound (or worsst case) on the running time of an algorithm with respect to its size parameters, while big-Ω is useful for providing a lower bound (or best case) on the algorithm's running time.