Counting Principles

Since computer scientists and engineers work with discrete sets of objects, counting and estimating set cardinalities represents a very important skill. In this lecture we review some basic counting techniques which most of us subconsciously tend to use on a regular basis.

Sum Rule

Sum Rule: let $A_1, A_2, \ldots, A_n$ be a pairwise disjoint (i.e. $A_i \cap A_j = \emptyset$, for all $i \neq j$) collection of sets. Then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i|.$$

Example 1. A juice shop offers 3 kinds of fruit juices, 2 vegetable juices, 10 smoothies, and 3 shakes. How many different kinds of drinks does the juice shop offer?

Example 2. How many pairs of positive numbers (i, j) satisfy the conditions that $1 \leq i \leq 10$ and $j|i$?

Product Rule

Product Rule: let $A_1, A_2, \ldots, A_n$ be a collection of sets. Then

$$|A_1 \times A_2 \times \cdots \times A_n| = \prod_{i=1}^{n} |A_i|.$$

Example 3. Let A be a set having five elements, and B a set with three elements. How many functions can be defined that have domain A and co-domain B?

Example 4. An alphabet contains 26 different letters (symbols). How many words (i.e. sequences of letters) of length 6 can be formed using this alphabet?

Example 5. How many different seven-digit phone numbers can be formed, assuming the first digit is neither 0 nor 1?

Example 6. When Ronald dresses, he has the option of wearing either one of his four pairs of blue jeans or one of his five pairs of slacks. When wearing blue jeans, he always wears one of five different casual shirts, and has the option of wearing one of three different jackets, and the option of wearing one of two different hats. When wearing slacks, he always wears one of four different dress shirts, and has the option of wearing one of three different ties, and the option of wearing a coat. How many different ways can Ronald dress himself? Assume that all casual shirts go with all jeans, all dress shirts go with all slacks, and his choice of jeans/slacks and shirt does not affect his choice of accessories (jacket and hats in case of jeans, tie and coat in case of slacks).

Principle of Inclusion-Exclusion

Principle of Inclusion-Exclusion: let A and B be two finite sets. Then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Example 7. How many n-bit binary strings either begin with 0 or end with 1?

Example 8. How many numbers between 1 and 100 (inclusive) are either divisible by 5 or 3?

Counting a set by counting its complement. Let A be a finite set and suppose $A \subset \mathcal{U}$, where $\mathcal{U}$ is a finite universal set. Then

$$|A| = |\mathcal{U}| - |\overline{A}|.$$

Example 9. How many 10-bit binary strings have at least one zero bit.

In the following example we make use of the inclusion-exclusion principle with three sets:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Example 10. How many 4-bit binary strings do not have two consecutive ones?

Pigeonhole Principle

Pigeonhole Principle: if $k+1$ pigeons are distributed into k holes, then at least one hole contains two pigeons.

Example 11. For a group of five students, at least two students have the same class level.

Example 12. If a drawer contains red, white, and blue socks, then randomly choosing four of the socks guarantees that two of the sock have the same color.

Generalized Pigeon-Hole Principle: if m pigeons are distributed into k holes, then there is one hole which contains at least $\lceil \frac{m}{k} \rceil$ pigeons.

Proof. Assume no hole contains $\lceil \frac{m}{k} \rceil$ pigeons. Then every hole contains at most $\lceil \frac{m}{k} \rceil - 1$ pigeons. Let $\epsilon \in [0, 1)$ be a number such that $\lceil \frac{m}{k} \rceil = m/k + \epsilon$. Then this gives at most $k(m/k + \epsilon - 1) = m - k + \epsilon k < m$ pigeons, which is a contradiction. Therefore, some hole must have at least $\lceil \frac{m}{k} \rceil$ pigeons.

Example 13. At a university with 33,000 students, there must be a day in which at least $\lceil \frac{33000}{366} \rceil = 91$ students have their birthday on that day.

Example 14. Given 65 calendar days, at least $\lceil \frac{65}{7} \rceil = 10$ of them fall on the same day of the week.

Permutations and Combinations

Given a set $S = \{s_1, s_2, \ldots, s_n\}$ with n members, an r-**permutation** of this set is is a sequence

$$(s_{i_1}, s_{i_2} \cdots s_{i_r})$$

of r distinct members of S.

For example if $S = \{a, b, c, d, e, f\}$ then (d, e, b), (f, e, d), (a, c, e), and (f, a, b) are some examples of 3-permutations of S.

Notice how we cannot use the product rule to count the number of r-permutations, since a permutation does not allow for a member to be repeated. Instead, the number of r-permutations of a set with n members is equal to

$$P(n, r) = n(n-1)\cdots(n-r+1),$$

since there are n ways to choose the first member, then $n-1$ ways to choose the second, all the way down to $n-r+1$ ways to choose the r th.

If $r = n$ (in other words, all n set members are ordered) then this n-permutation is simply called a **permutation** for short.

Example 15. Suppose 3 people get on an elevator in a six-story building and each one gets off on a different floor. How many ways can this be done?

Example 16. How many one-to-one functions can be specified from a domain of 4 elements to a co-domain of 6 elements? In general, how many one-to-one functions are there from a domain of m elements to a co-domain of n elements?

Note that when $r = n$, we have $P(n, n)$ equals the number of ways of ordering n members from a set with n members. When all members of the set are being ordered we simply call this a **permutation** on n elements. Also notice that

$$P(n, n) = n(n-1)(n-2)\cdots 2 \cdot 1 = n!.$$

Combinations

Given a set $S = \{s_1, \ldots, s_n\}$ with n members, an r-**combination** (also called an r-**subset**) of S is a subset of S of size r. Similar to a permutation, an r-combination requires selecting r members from S, but a combination differs in that the members are not ordered, but rather placed in a set.

The number of ways to choose r members from a set having n members is

$$C(n,r) = \frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{n!}{(n-r)!r!}.$$

A more common way of writing $C(n,r)$ is

$$\binom{n}{r}$$

Notice that $C(n,r) = P(n,r)/r!$. That is because there are $r!$ different sequences that can be made from a subset of r members. Therefore, there is an $r!$ to 1 ration between r-permuations and r-combinations, and so $P(n,r)$ must be divided by $r!$ in order to obtain the number of r-subsets.

Example 17. Enumerate all three permutations of the set $\{a, b, c, d\}$ and examine why we must divide $P(4, 3)$ by $3!$ in order to get the total number of three combinations.

Example 18. How many ways can a department with 20 graduate students form a softball team with 9 players with prescribed positions for each player? with non-prescribed positions for each player?

Example 19. Same question as the previous example, but now assume that half the students are women, and that the team must have at least 4 women.

$C(n, r)$ is called a **binomial coefficient** since, in the expansion of $(x + y)^n$, the coefficient of the $x^{n-r}y^r$ term is $C(n, r)$.

Example 20. What is the coefficient of the x^4 term in the expansion of $(3x + 2)^6$?

Example 21. The Birthday problem. What is the probability that two people in this classroom have the same birthday?

Theorem 1. Some combination identities

1. $C(n, r) = C(n, n-r)$
2. $\sum_{k=0}^{n} C(n, k) = 2^n$
3. $\sum_{k=0}^{n} (-1)^k C(n, k) = 0$
4. **Pascal's Identity:** $C(n, k) = C(n-1, k) + C(n-1, k-1)$
5. **Vandermonde's Indentity:** let $k \leq m, n$. Then

$$C(m+n, k) = \sum_{r=0}^{k} C(m, r)C(n, k-r)$$

Example 22. Prove the above identities.

Example 22 Continued.

General Combinations

Given a set of n elements, A k**-general-combination** is a k-subset of the set, in which elements are allowed to be repeated. Such a subset is called a **multiset**. For example, some 3-multisets of the set $\{1, 2, 3, 4\}$ include $\{1, 3, 4\}$, $\{2, 2, 3\}$, and $\{4, 4, 4\}$.

Example 23. List all 2-multisets of the set $\{1, 2, 3\}$.

Theorem 2. The number of k-multisets of a set of n elements is $C(n+k-1, k)$.

Proof of Theorem 2. To count the number of k-multisets, one adds $k-1$ more elements to the set of n elements. These new elements are called **repeaters** and are denoted as $r_2, r_3, \ldots, r_k$. Here r_i indicates that the element listed in component $i-1$ of the sequence will be repeated in component i. Whenever, an element is repeated in the multiset, it means that one of the repeaters was chosen. For example, the set $\{5, 5, 5, 5, 5\}$ corresponds with choosing $5, r_2, r_3, r_4, r_5$, while the set $\{5, 5, 5, 6, 6\}$ corresponds with choosing $5, r_1, r_2, 6, r_4$. Moreover, there are exactly $C(n+k-1, k)$ ways to choose a k-subset of elements and repeaters. And for each choice, there is a natural way to convert it to a unique k-multiset. Finally, each k-multiset of elements can be represented as a choice of elements and repeaters. Hence, there is a one-to-one correspondence between the $C(n+k-1, k)$ k-subsets of elements and repeaters, and k-multisets of elements.

Example 24. Express the 6-multiset $\{1, 2, 2, 3, 3, 3\}$ as a 6-subset of elements and repeaters. Given the 7-subset of elements and repeaters $\{5, 8, 9, r_3, r_5, r_6, r_7\}$, provide the multiset that is associated with this set.

Example 25. An academic department consisting of 15 professors must send a representative to each of five different university committees. How many ways can this be accomplished if a professor can serve on multiple committees?

Example 26. A k**-dimensional monomial of degree** n is a product of variables of the form $x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$, where $n_1 + n_2 + \cdots + n_k = n$. Provide some examples of 4-dimensional monomials of degree 8. How many 4-dimensional monomials of degree 8 are there? How many k-dimensional monomials of degree n are there? How many k-dimensional monomials of degree less than or equal to n are there?

Generalized Product Rule

Generalized Product Rule. Suppose there are m rounds, and in each round a selection is made. Moreover, let n_i denote the number of options that are available in Round i, $i = 1, \ldots, n$, and suppose that this number does not depend on the selections made in previous rounds. Then the number of distinct selection sequences that can be realized is equal to

$$n_1 \times n_2 \times \cdots \times n_m.$$

Example 27. How many ten-letter words over the alphabet $\{a, b, c\}$ have the property that no two consecutive characters are the same? For example, the string "abcbcbabcb" would count and the strings "abbbcbabcb" and "aacbcbabcb" would not count.

Example 28. A license plate for some state is formed using seven characters: the first character being a digit (0 through 9), the next four being capital letters (A through Z) and the last two being digits.

a. How many different license platea are possible?

b. How many license plates are possible if no digit appears more than once?

c. How many license plates are possible if no digit or letter appears more than once?

Exercises

1. An office building has 27 floors and 37 offices on each floor. How many offices does the building have? What counting rule did you use? Provide n, the number of sets and define each A_i set, $i = 1, 2, \ldots n$.

2. A multiple-choice test has 10 questions. Each question has four choices. How many different ways can a student respond to the test? What if students are allowed to leave an answer blank? What counting rule did you use? Provide n, the number of sets and define each A_i set, $i = 1, 2, \ldots n$.

3. A particular type of shirt comes in 5 different colors, has 4 different sizes, and has both a male and female version. How many different kinds of shirts can be ordered? What counting rule did you use? Provide n, the number of sets and define each A_i set, $i = 1, 2, \ldots n$.

4. There are four major auto routes from Boston to Detroit, and six major routes from Detroit to Los Angeles. How many major routes are there from Boston to Los Angeles via Detroit? What counting rule did you use? Provide n, the number of sets and define each A_i set, $i = 1, 2, \ldots n$.

5. How many different 3-letter initials can people have? For example JFK is one such 3-letter initial. What counting rule did you use? Provide n, the number of sets and define each A_i set, $i = 1, 2, \ldots n$.

6. A state used to issue seven-digit license plates, but recently changed to five-letter plates. Assuming the seven-digit plates have not been removed from circulation, how many different plates can be issued (past and present) ?

7. How many bit strings are there of length six or less (including the empty string)?

8. How many numbers between 1 and 1000 are divisible by i) 7 and 11, ii) 7 but not 11, iii) 11 but not 7, iv) 7 or 11, v) neither 7 nor 11?

9. How many numbers between 100 and 500 (inclusive) are divisible by i) 2 and 3, ii) 2 but not 3, iii) 3 but not 2, iv) 2 or 3, v) neither 2 nor 3?

10. How many numbers between 1 and 1000 (inclusive) are divisible by either, 2, 3, or 5?

11. How many strings of three decimal digits i) do not contain the same digit 3 times, ii) begin with an odd digit, iii) have exactly two digits that are 4's?

12. How many license plates can be made that either begin with 3 letters followed by 3 digits, or with 3 digits followed by 3 letters?

13. How many strings of eight English capital letters are there i) if letters can be repeated, ii) if no letter can be repeated, iii) that start with X if letters can be repeated, iv) that end with X if letters can be repeated?

14. How many different functions can be defined from a set with 10 elements, to a set with 4 elements? How many different functions can be defined from a set with m elements, to a set with n elements?

15. How many different one-to-one functions can be defined from a set with 10 elements, to a set with 15 elements? How many different functions can be defined from a set with m elements, to a set with n elements, where $m \leq n$?

16. A palindrome is a string whose reversal is identical to the string. How many bit strings of length n are palindromes, assuming n is odd?

17. How many bit strings of length seven either begin with two zeros or end with three ones?

18. There are ten people at a wedding, including the bride and groom. How many ways can six of those people be arranged in a row, if i) both the bride and groom must be included among the six, ii) only the bride is included among the six, iii) the bride is not next to the groom, iv) both the bride and groom must be together, v) the bride is standing somewhere to the left of the groom?

19. Show that if there are 27 students in class, then at least two of them must have last names that begin with the same letter.

20. A drawer has a dozen brown socks, and a dozen black socks, all unmatched. A man takes socks out of the drawer at random in the dark. What is the minimum number of socks he must take out to be sure that he has a matching pair? What is the minimum number of socks he must take out in order to ensure that he has a matching pair of black socks?

21. Show that for any set of five integers, there must be at least two that have the same remainder when divided by four.

22. Show that for any set of $d+1$ integers, there must be at least two that have the same remainder when divided by d.

23. Show that any function with a finite domain of m elements, and a finite co-domain of n elements cannot be one-to-one if $m > n$.

24. Let (x_i, y_i), $i = 1, 2, 3, 4, 5$ be a set of five distinct points with integer coordinates in the xy-plane. Consider all the line segments that can be formed that have two of these points as the end points. Show that at least one of these line segments must have a midpoint that has integer coordinates.

25. Let (x_i, y_i, z_i), $i = 1, 2, \ldots, 9$ be a set of nine distinct points with integer coordinates in the xyz-plane. Consider all the line segments that can be formed that have two of these points as the end points. Show that at least one of these line segments must have a midpoint that has integer coordinates.

26. How many 5-permutations are there of the set $\{a, b, c, d, e, f, g\}$. Of these 5-permutations, how many end in g?

27. Compute i) $P(8, 5)$, ii) $P(5, 5)$, iii) $P(100, 1)$.

28. How many different ways can five runners finish a race if no ties are allowed?

29. Starting at the ground floor, how many different ways can four people get off on different floors of an elevator for a seven-story building?

30. How many permutations of the letters ABCDEFG contain i) the string BCD, ii) the string CFGA, iii) either the string BCD or the string CFGA? Hint: for example, it might help to think of BCD as a single letter.

31. How many permutations of the letters ABCDEFGH contain i) the strings AB, DE and GH, ii) the strings BA and FGH, iii) the strings BCA and ABF?

32. How many ways are there to place eight men and five women in a line, if no two women are to stand next to each other?

33. One hundred tickets $1, 2, \ldots, 100$ are sold to one hundred different people. Four different prizes are awarded, including a grand prize. How many different ways are there to award the prizes if i) there are no restrictions, ii) the person holding ticket 47 wins one of the prizes. iii) The people holding tickets 19 and 47 both win prizes. iv) the grand prize winner is a person holding one of the tickets: 2, 35, 47, or 78. v) The people holding tickets 2 and 35 win prizes, but the people holding 47 and 78 don't.

34. Compute the following i) $C(9, 4)$, ii) $C(20, 10)$, iii) $C(20, 17)$

35. A club has 25 members. How many ways are there to choose an executive committee of four members?

36. How many 4-permutations of the first 100 positive integers have a subsequence of three consecutive integers i) if the subsequence is contiguous (examples include 8,9,10,51, and 2,5,6,7), ii) if the subsequence is not necessarily contiguous (examples include 5,18,6,7 and 56,57,88,58).

37. Seven women and nine men are in a gym class. How many ways can four team captains be chosen if two must be from each gender?

38. How many strings of six lowercase English letters contain i) exactly one vowel, ii)at least two vowels? Note: there are 5 vowels and 21 consonants.

39. Find the coefficient of x^8y^9 is $(3x + 2y)^{17}$.

40. Find the coefficient of x^k in the expansion of $(x^2 - 1/x)^{100}$.

41. Draw the first eleven rows (up to $n = 10$) of Pascal's triangle.

42. Verify Vandermonde's identify for $m = 5$, $n = 4$, and $k = 3$.

43. How many 5-multisubsets are there of a set with 10 objects?

44. A bagel shop has three different kinds bagels. How many ways can one choose a dozen bagels from the shop?

45. Provide the multiset that corresponds with the selection i) $\{3, 5, 9, r_2, r_3, r_6\}$, ii) $\{6, r_2, r_3, r_4\}$, iii) $\{1, 5, 6, 7, r_5\}$

46. Write each multiset as a non-multiset that may include repeaters i) $\{2, 4, 4, 5, 5\}$, ii) $\{1, 1, 1\}$, iii) $\{1, 2, 2, 3, 3, 3\}$

47. How many 10-bit binary strings either have five consecutive 1's or five consecutive 0's? Hint: letting A denote the 10-bit binary strings having five consecutive 1's, apply the sum rule to A by breaking it into 6 non-overlapping cases.

48. A password must have exactly six characters chosen from digits, lowercase letters, and four special characters. How many passwords can be formed if no character can be repeated? How many passwords can be formed if the first character cannot be a special character?

49. A manager must select three coders from her group to write three different software projects. There are 7 junior and 3 senior coders in her group. The first project can be written by any of the coders. The second project must be written by a senior person and the third project must be written by a junior person. How many ways are there for her to assign the three coders to the projects if no person can be assigned to more than one project?

Exercise Solutions

1. Use sum or product rule. 999
2. Use product rule. i) 1048576, ii) 9765625
3. Use product rule. 40
4. Use product rule. 24
5. Use product rule. 17576
6. $10^7 + 26^5$
7. $1 + 2 + 2^2 + \cdots + 2^6 = 2^7 - 1 = 127$
8. Use inclusion exclusion principle and complement rule. i) 12, ii) 130, iii) 78, iv) 220, v) 780
9. Use inclusion exclusion principle and complement rule. i) $83 - 16 = 67$, ii) $201 - 67 = 134$, iii) $133 - 67 = 66$, iv) $201 + 133 - 67 = 267$, v) $401 - 267 = 134$.
10. Use inclusion exclusion principle for three sets. 734
11. i) Use complement rule. 990, ii) Use sum rule where A_i elements begin with odd digit i. 500, iii) Use product rule: first place the two fours (there are three ways to do this); then fill in the remaining digit (9 ways). 27
12. Use sum rule where both $|A_1|$ and $|A_2|$ can be found using the product rule. $2(10^3 26^3)$.
13. i) Use product rule. 208827064576, ii) $P(26, 8)$, iii) Use product rule, but now $|A_1| = 1$. 8031810176, iv) Same as iii, but now $|A_8| = 1$.
14. i) Use product rule. 1048576, ii) Use product rule. n^m.
15. i) $P(15, 10)$, ii) $P(n, m)$
16. Use product rule. 2^{k+1}, where $n = 2k + 1$. The first $k + 1$ bits completely determine the rest of the bit string.
17. Use inclusion-exclusion principle. 44
18. i) Use product rule (first place the bride and groom, then place four guests). $P(6, 2)P(8, 4)$, ii) Use product rule (first place the bride, then place five guests). $P(6, 1)P(8, 5)$, iii) Use product rule (first place the bride and groom, then place four guests). $(P(6, 2) - 10)P(8, 4)$, iv) Use product rule (first place the bride and groom, then place four guests). $10P(8, 4)$, v) Use product rule (first place the bride and groom, then place four guests). $(1 + 2 + \cdots + 5)P(8, 4)$.
19. Use PHP: letters are holes, students are pigeons.
20. i) 3, ii) 14
21. Use PHP: remainders are holes, numbers are pigeons.
22. Use PHP: remainders are holes, numbers are pigeons.

23. Use PHP: co-domain elements are holes, domain elements are pigeons.

24. Use PHP: parity combinations (e.g. (ODD,EVEN), (EVEN,ODD)) of components are holes, points are pigeons.

25. Use PHP: parity combinations (e.g. (ODD,EVEN,EVEN), (EVEN,EVEN,ODD)) of components are holes, points are pigeons.

26. i) $P(7,5)$, ii) $P(6,4)$

27. i) 6720, ii) 120, iii) 100

28. $P(5,5)$

29. $P(6,4)$

30. i) $P(5,5)$, ii) $P(4,4)$, iii) Use sum rule. $P(5,5)+P(4,4)$.

31. i) $P(5,5)$, ii) $P(5,5)$, iii) 0

32. First order the men: $P(8,8)$. There are now nine places to place a woman (seven in-between two men and two more at the front and end of the line). Now order the women into those nine places: $P(9,5)$. Finally, use product rule: $P(8,8)P(9,5)$, since ordering the men is independent of how the women are ordered.

33. i) $P(100,4)$, ii) First assign a prize to 47, and then assign the remaining prizes to the remaining 99 people: $4P(99,3)$, iii) $P(4,2)P(98,2)$. iv) $4P(99,3)$, v) $P(4,2)P(96,2)$.

34. i) 126, ii) 184,756, iii) 1140

35. $C(25,4)$.

36. i) Let A denote the set of 3-permutations of $1,\ldots,100$ that have the form $(x,x+1,x+2,y)$ where $1 \le x \le 98$, and y is any other integer between 1 and 100. Similarly, B denots the set of 3-permutations of $1,\ldots,100$ that have the form $(y,x,x+1,x+2)$ where $1 \le x \le 98$, and y is any other integer between 1 and 100. Thus, $|A|=|B|=(97)(98)$. Moreover, $A\cap B$ is the set of 3-permutations having the form $(x,x+1,x+2,x+3)$, where $1 \le x \le 97$. Thus, $|A\cap B|=97$. Therefore, by the principle of inclusion-exclusion, we have the set of 3-permutations that possess at least 3 consecutive numbers is $A\cup B$, and

$$|A\cup B| = 2(97)(98) - 97 = 18915.$$

ii) Similar to i), but now use four sets: $A:(y,x,x+1,x+2)$, $B:(x,y,x+1,x+2)$, $C:(x,x+1,y,x+2)$, and $D:(x,x+1,x+2,y)$. Notice that only A and D can overlap, and so the principle of inclusion-exclusion for two sets can be used:

$$|A\cup B\cup C\cup D| = 4(97)(98) - 97 = 37927.$$

37. Use the product rule: $C(7,2)C(9,2)$.

38. i) Choose the vowel; then choose its location; then fill in the remaining letters with consonants: $(5)(6)(21^5)$, ii) Use complement rule: $26^6 - 21^6 - (5)(6)(21^5)$

39. $2^9 3^8 C(17,8)$

40. For each k determine the value of j such that $x^k = (x^2)^j (1/x)^{100-j}$. The coefficient is then $C(100,j)(-1)^{-j}$

41. Use Pascal's Identity.

42. 84

43. $C(14,5)$

44. $C(14,12)$

45. Provide the multiset that corresponds with the selection i) $\{3,3,3,5,9,9\}$, ii) $\{6,6,6,6\}$, iii) $\{1,5,6,7,7\}$

46. i) $\{2,4,5,r_3,r_5\}$, ii) $\{1,r_2,r_3\}$, iii) $\{1,2,3,r_3,r_5,r_6\}$

47. Let A denote the 10-bit binary strings having five consecutive 1's. Consider the following subsets of A.

$$A_0 = 111110XXXX$$

For example, A_0 is the set of all binary strings that begin with five 1's followed by a 0, and end with any four-bit binary string. Similarly,

$$A_1 = X111110XXX,$$

$$A_2 = XX111110XX,$$

$$A_3 = XXX111110X,$$

$$A_4 = XXXX111110,$$

and

$$A_5 = XXXXX11111.$$

It's an exercise to verify that all of the above sets are non-overlapping. Then

$$|A| = \sum_{i=0}^{5} |A_i| = 16 + 16 + 16 + 16 + 16 + 32 = 112.$$

Similarly, if B denotes the 10-bit binary strings having five consecutive 0's, then $|B| = 112$. Moreover $|A \cap B| = 2$ since

$$A \cap B = \{0000011111, 1111100000\}.$$

Therefore, by the principle of inclusion-exclusion, there are $2(112) - 2 = 222$ such binary strings.

48. How many passwords can be formed if no character can be repeated? Anwer: $P(40,6)$ since the there are 40 possible characters, and the password will form a 6-permutation over this set of characters. How many passwords can be formed if the first character cannot be a special character? Answer: $36 \cdot (40)^5$, since there are 36 choices for the first character, and 40 for the remaining five.

49. We need to start off with cases since the choice for Project 1 may affect the number of choices for Projects 2 and 3. Case 1: Project 1 is assigned a senior programmer. Then we have $3 \times 2 \times 7 = 42$ possible ways to cover the projects. Case 2: Project 1 is assigned a junior programmer. Then we have $7 \times 3 \times 6 = 126$ ways to cover the projects. Therefore, there are $42 + 126 = 168$ possible ways to cover the projects.