Functions

Functions are considered the workhorses of any programming language, because they often serve the purpose of transforming data into other (perhaps more useful) data. For example, suppose that on a given day a package-delivery company has 97 working delivery trucks, 136 able and willing drivers (each with a unique preferred work schedule), and 8,643 packages that need delivering by various times and to various residential and business addresses throughout some region. What this company needs is a function that can input this data, and output a delivery route for each truck, a work schedule for each driver, and an assigned truck for each driver so that all the packages are delivered on time. Also, functions form the building blocks of expressions, which in turn can be used to create formal languages, such as the one used in this book.

Let x be a variable whose domain set is A, and let B be a set. Then a **function** f of a single variable x into B represents a means for assigning every $a \in A$ with exactly one $b \in B$. The following is some terminology and notation that is used to describe functions.

- A is called the **domain** of f.
- B is called the **codomain** of f.
- $f : A \to B$ is a notation that indicates f is a function that assigns members of set A to members of set B.
- $f(x)$ is a common notation that is used when both the domain of x (and hence f) and the codomain of f are already understood.
- $f(a) = b$ indicates that f has assigned $a \in A$ to $b \in B$, where b is called the **image** of a under f, and a is called a **preimage** of b under f. It is also common to call a the **input** and $f(a) = b$ its assigned **output**.

Like variables, every function has a name, and it is common to use generic names, such as f, g, and h, when speaking of some arbitrary (i.e. no one in particular) function, just as it is common to use names like x, y, and z to represent an arbitrary variable.

Example 1. Consider the function $\text{grade}(s)$ that assigns student s a letter grade from the set $G = \{a, b, c, d, f\}$, where

$$\text{dom}(s) = S = \{\text{Ann}, \text{Ethan}, \text{Jaspinder}, \text{Pam}\}.$$

Thus, S is the domain of `grade`, and G is its codomain. Finally the student grades assigned by `grade` are $\text{grade}(\text{Ann}) = a$, $\text{grade}(\text{Ethan}) = c$, $\text{grade}(\text{Jaspinder}) = c$, and $\text{grade}(\text{Pam}) = b$.

Example 2. Let S denote the set of students attending a university, I the set of instructors at the university, and s a variable whose domain is S. For a student s attending a university, let $\text{ins}(s)$ denote the instructor of the first calculus course that s enrolled in at the university. Then $\text{ins} : S \to I$ is not a function because not every student (e.g. a student majoring in music) will elect to take calculus at the university. A function must assign *every* member of the domain to some member of the codomain. $\square$

Example 3. Consider the following description of a function $f(x)$. For positive real number x, $f(x)$ is the real number whose square equals x. Is this a valid function? On the surface it may appear that way, but consider $x = 4$. From the description $f(x)$ is the real number whose square equals 4. There are actually two such numbers: 2 and -2, but it's not valid to say $f(4) = \pm 2$ since, by definition, a function assigns each input x to *exactly* one member in the codomain which in this case is $\mathcal{R}$. On the other hand, had the description read "$f(x)$ is the *positive* real number whose square equals x", then there is no ambiguity about the output, namely $\sqrt{x}$, to which x is being assigned. In mathematics, a **relation** is a structure that allows an input to be assigned to more than one output. $\square$

Example 4. The **Twin Prime Conjecture** asserts that there are infinitely many pairs of prime numbers that differ by 2. Examples of such pairs include, 2 and 3, 3 and 5, and the pair 11 and 13. At this writing no mathematician has been able to prove or disprove this conjecture. Now consider $f : \mathcal{N} \to \{0, 1\}$, where, for all $n \in \mathcal{N}$, we define $f(n) = 1$ if the Twin Prime Conjecture is true. Otherwise, $f(n) = 0$. Is f a valid function?

Example Solution.

Describing a Function

When the domain of a function $f(x)$ is finite and not too large, then the most common way of describing f is to list each $f(a) = b$ pair, as was done in Example 1. On the other hand, when the domain is infinite or very large, then it becomes necessary to use a rule to describe the relationship between the input and output. Such rules are usually algebraic, in that they are described with an algebraic expression that depends on the input variable x. For example, if x has the domain of all real numbers, then $f(x) = 3x^2 + 7x - 5$ provides a rule for assigning an arbitrary domain input to a codomain output. Indeed, x represents an arbitrary domain input, and the rule states that x should be assigned to codomain value $3x^2 + 7x - 5$. For example, if $x = -2$, then $f(-2) = 3(-2)^2 + (7)(-2) - 5 = -7$. This is an example of a **real-valued function**, meaning that both its domain and codomain is equal to the set of real numbers $\mathcal{R}$. In fact, the main goal of calculus is to study the properties of real-valued functions.

Example 5. The **floor** operation, denoted $\lfloor x \rfloor$ appears often in function rules, where $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to real number x. Some examples include $\lfloor 3.2 \rfloor = 3$, $\lfloor 7 \rfloor = 7$, $\lfloor 0.341 \rfloor = 0$, and $\lfloor -2.45 \rfloor = -3$. Similarly, the **ceiling** operation, denoted $\lceil x \rceil$ is defined as the least integer that is greater than or equal to x. Some examples include $\lceil 3.2 \rceil = 4$, $\lceil 7 \rceil = 7$, $\lceil 0.341 \rceil = 1$, and $\lceil -2.45 \rceil = -2$. □

In practice, many functions require far more complex rules than what can be provided by a single algebraic expression. Such rules are usually described with a procedural programming language, such as Python or C++. Although we do not study this approach here, in a later chapter we describe how to use a *constraint* program to define the relationship between a function's input and output.

Example 6. Let $\text{dom}(n) = \mathcal{N}^+$ and $f(n)$ denote the n th term in the sequence $-1, 0, 3, 8, 15, 24, 35, 48, \ldots$. For example, $f(1) = -1$, $f(2) = 0$, etc.. Provide an algebraic rule for computing f that only uses variable n.

Example Solution.

Properties of Functions

The **range** of a function $f : A \to B$, denoted range(f) is the set of all outputs of function f. Thus, range(f) $\subseteq B$.

Example 7. For the `grade` function from Example 1, we see that range(grade) $= \{a, b, c\}$ since one student was assigned an a, one a b, and two were assigned c. On the other hand no one was assigned d or f, and so $d, f \notin$ range(grade). Similarly, range($\lfloor x \rfloor$) $= \mathcal{I}$, since the floor function only outputs integers, and every integer serves as an output for some real input, e.g. $\lfloor n \rfloor = n$ for every $n \in \mathcal{I}$.

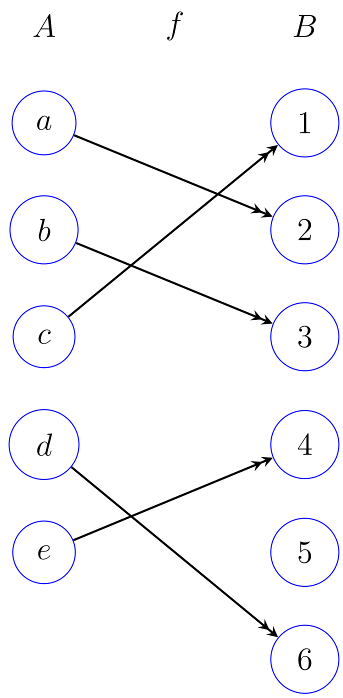

Figure 1: Example of a one-to-one function

Properties of functions: let $f : A \to B$ be a function. Then f is

one-to-one iff for every $b \in B$ there is at most one $a \in A$ such that $f(a) = b$.

onto iff for every $b \in B$ there is at least one $a \in A$ such that $f(a) = b$.

one-to-one correspondence iff it is both one-to-one and onto.

Thus, a function $f : A \to B$ is one-to-one provided every member $b \in B$ has at most one domain member assigned to it, and f is onto if and only if $\text{range}(f) = B$.

Figures 1 and 2 show examples of one-to-one and onto functions, respectively. For example, an arrow from the node labeled with a to the node labeled with 2 means that $f(a) = 2$.

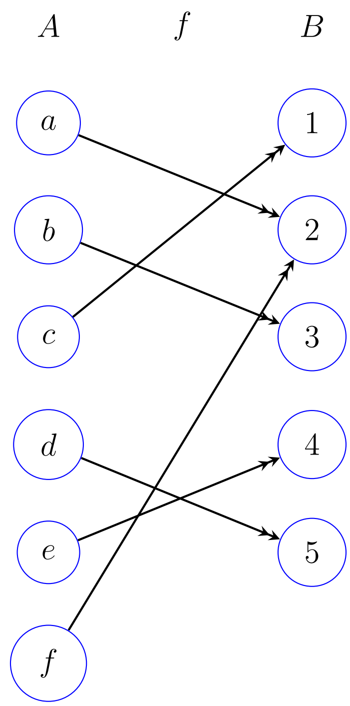

Figure 2: Example of an onto function

Example 8. Consider the function $f : \mathcal{N} \rightarrow S$, where S is the set of all bit strings, such as 0, 100, 010111, etc.. Moreover, $f(n) = s$, where s is the shortest bit string that represents n as a binary number. For example, $f(6) = 110$ since $1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 = 6$. First notice that f is a well-defined function since every natural number can be written in binary, and there is a unique smallest bit string whose decimal value equals n. Also, f is one-to-one since every bit string s has a unique decimal value, say n, and so *at most* one natural number, namely n, can be assigned to s. However, f is not onto since not every bit string has a preimage. For example, $s = 00011001$ has an associated decimal value of $n = 25$, but $f(25) = 11001 \neq 00011001$, since 11001 and 00011001 are different strings. □

Example 9. Consider the previous example, but now interchange the domain and codomain, meaning that we now have a function $f : S \to \mathcal{N}$, where f assigns each bit string s the integer n, where s is a valid binary representation of n. For example, $f(001101) = 13$, since

$$13 = 0 \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0.$$

Again, notice that f is well-defined, since every bit string is a binary representation for some unique $n \in \mathcal{N}$. Secondly, f is also onto, since every $n \in \mathcal{N}$ has a preimage s, i.e. $f(s) = n$, since every $n \in \mathcal{N}$ has a binary representation. Finally, f is *not* one-to-one, since every $n \in \mathcal{N}$ has infinitely many preimages. In fact if $f(s) = n$, then $f(0^m \cdot s) = n$ as well, where $0^m \cdot s$ is the bit string obtained by placing m zeros in front of s, for any $m \geq 0$. On the other hand, being one-to-one means that every codomain member is allowed at most one preimage. $\square$

Images and preimages of sets

Let $f : A \to B$ be a function. Then for $b \in B$, $f^{-1}(b) \subseteq A$ denotes the set of preimages of b with respect to f.

Example 10. For the floor function $f(x) = \lfloor x \rfloor$, $f^{-1}(2)$ denotes the real numbers x for which $\lfloor x \rfloor = 2$. But this is all real numbers between 2 and 3, not including 3. Thus, using interval notation from calculus, we have $f^{-1}(2) = [2, 3)$, where the left bracket indicates that 2 is included in the set, while the right parenthesis indicates that 3 is not included. Similarly, for $f(x) = \lceil x \rceil$, $f^{-1}(2) = (1, 2]$, where the left parenthesis indicates that 1 is not in the set, since $f(1) = 1 \neq 2$.

Example 11. For the function $f : S \to \mathcal{N}$ from Example 9, we have

$$f^{-1}(6) = \{110, 0110, 00110, \ldots\}$$

which is the set of all bit strings that begin with m 0's, followed by 1100, where $m \geq 0$.

Given $f : A \to B$, now let $S \subseteq A$ and $T \subseteq B$ be subsets of the domain and codomain respectively. We have the following definitions.

f-image of S denoted as $f(S)$, and is defined as the union of all images $f(s)$, where $s \in S$ is arbitrary.

f^{-1}-preimage of T denoted as $f^{-1}(T)$, and is defined as the union all of preimages $f^{-1}(t)$, where $t \in T$ is arbitrary.

Example 12. For the function $f(x) = \lfloor x \rfloor$, $f([-3.5, 2.8])$ is the set of all images of real numbers between -3.5 and 2.8. But the floor of all of these numbers falls between -4 and 2. Therefore, $f([-3.5, 2.8]) = \{-4, -3, -2, -1, 0, 1, 2\}$.

Example 13. For the function $f : \mathcal{N} \to S$ from Example 8, $f^{-1}(\{00, 01, 10, 11\})$ denotes set of natural numbers n for which $f(n) \in \{00, 01, 10, 11\}$. And there are only two such n, namely 2 and 3 (why?). Therefore, $f^{-1}(\{00, 01, 10, 11\}) = \{2, 3\}$.

Operations on functions

Let f and g be two functions, both having the same domain A and codomain B. Notice that whatever operations that may be performed on members of B may also be performed on f and g. For example, suppose $B = \mathcal{R}$. In this case we may perform arithmetic operations on members of B, since these members are real numbers. But this in turn allows us to perform arithmetic operations on f and g. For example, since we may add two real numbers b_1 and b_2 to get a third real number $b_1 + b_2$, so we may add f and g to get a third function, $f + g$, whose domain is A, codomain is $B = \mathcal{R}$, and whose rule is $(f + g)(a) = f(a) + f(b)$. The following definition summarizes the most common arithmetic operations on functions.

Definition 1. Let $f, g : A \to B$ be functions, and suppose $B \subseteq \mathcal{R}$ is a set of real numbers that is closed under addition, subtraction, multiplication, and division (e.g. B is closed under addition iff $b_1, b_2 \in B$ implies $b_1 + b_2 \in B$). Then the following functions are well-defined for all $a \in A$.

Addition $(f + g)(a) = f(a) + g(a)$

Subtraction $(f - g)(a) = f(a) - g(a)$

Multiplication $(fg)(a) = f(a)g(a)$

Division $(f/g)(a) = f(a)/g(a)$ (Exception: $(f/g)(a)$ is undefined whenever $g(a) = 0$)

Example 14. Let f and g be real-valued functions having rules $f(x) = 2x + 5$, and $g(x) = x^2 - 3$. Then the rules for $f + g$, $f - g$, fg, and f/g are $(f + g)(x) = (2x + 5) + (x^2 - 3) = x^2 + 2x + 2$,

$$(f - g)(x) = (2x + 5) - (x^2 - 3) = -x^2 + 2x + 8,$$

$$(fg)(x) = (2x + 5)(x^2 - 3) = 2x^3 + 5x^2 - 6x - 15,$$

and $(f/g)(x) = (2x+5)/(x^2-3)$. Futhermore, since $f(2) = (2)(2)+5 = 9$, and $g(2) = 2^2 - 3 = 1$, we have $(f + g)(2) = 9 + 1 = 10$, $(f - g)(2) = 9 - 1 = 8$, $(fg)(2) = (9)(1) = 9$, and $(f/g)(2) = 9/1 = 9$. Of course, we should get the same answers by using the rules for $f + g, \ldots, f/g$. For example, $(f + g)(2) = 2^2 + 2(2) + 2 = 10$. We leave it to the reader to verify this for $f - g, fg$, and f/g. □

In a similar manner to how arithmetic operations were defined for functions, we may also define set operations on functions having codomains that are closed under set operations. In what follows we let $\mathcal{S}$ denote the set of all possible sets (notice how $\mathcal{S}$ is a member of itself!).

Definition 2. Let $f, g : A \to B$ be functions, and suppose $B \subseteq \mathcal{S}$ is a set of sets that is closed under union, intersection, subtraction, symmetric difference, and complement. Then the following functions are well-defined for all $a \in A$.

Union $(f \cup g)(a) = f(a) \cup g(a)$

Intersection $(f \cap g)(a) = f(a) \cap g(a)$

Subtraction $(f - g)(a) = f(a) - g(a)$

Symmetric Difference $(f \oplus g)(a) = f(a) \oplus g(a)$

Complement $\overline{f}(a) = \overline{f(a)}$

Example 15. Let $A = \mathcal{N}$, and B denote the power set of all prime numbers. Then $f : A \to B$ is defined so that $f(n)$ is the set of all prime factors of n; i.e. those prime numbers that divide evenly into n. For example, since $24 = 2^3 \cdot 3^1$, we have $f(24) = \{2, 3\}$. Also, let $g : A \to B$ be defined so that $g(n)$ is the set of prime numbers p for which $\sqrt{n} \le p \le n$. Then $f(26) = \{2, 13\}$, $g(26) = \{7, 11, 13, 17, 19, 23\}$,

$$(f \cup g)(26) = \{2, 13\} \cup \{7, 11, 13, 17, 19, 23\} = \{2, 7, 11, 13, 17, 19, 23\},$$

$$(f \cap g)(26) = \{2, 13\} \cap \{7, 11, 13, 17, 19, 23\} = \{13\},$$

$$(f - g)(26) = \{2, 13\} - \{7, 11, 13, 17, 19, 23\} = \{2\},$$

$$(f \oplus g)(26) = \{2, 13\} \oplus \{7, 11, 13, 17, 19, 23\} = \{2, 7, 11, 17, 19, 23\},$$

and $\overline{f}(26)$ is the set of all prime numbers, with the exceptions of 2 and 13. □

Composing Two Functions

Let $f : A \to B$ and $g : B \to C$ be functions. Then one can define the composition of f with g, denoted as $(g \circ f) : A \to C$, where

$$(g \circ f)(a) = g(f(a)).$$

Example 16. Set $F = \{\text{salmon,halibut,mackeral}\}$ consists of types of fish that are unloaded off a boat. Set $T = \{a, b, c\}$ consists of totes that the respective fish are placed in. $p : F \to T$ is a function whose effect is to place a fish in a tote. Set $L = \{\text{coldstorage, cannery}\}$ is a set of locations where the totes are sent. $m : T \to L$ is a function whose effect is moving a tote from T to a location in L. Describe in words the effect of the composition of p with m, namely $m \circ p$. Evaluate $(m \circ p)(\text{salmon})$, $(m \circ p)(\text{mackeral})$, and $(m \circ p)(\text{halibut})$.

Example 17. Let $f(n)$ be the function from Example 15, and g be a function that takes as input a finite set s of natural numbers, and outputs the sum of its members. Given definitions for both $f \circ g$ and $g \circ f$. Compute $(f \circ g)(\{3, 5, , 6, 9\})$ and $(g \circ f)(36)$.

Example Solution.

Functions of several variables

So far we have focused entirely on functions of a single variable, but most functions in practice involve functions that depend on several variables. For example, a student's final grade in a course is often a (linear) function of several different assessments of competency, including those for written exams, projects, and research papers. Furthermore, in the era of "big data" it is not uncommon to define functions based on hundreds or even thousands of variables, where the rules of such functions are obtained via machine-learning algorithms. Some practical uses of such functions include recognizing sounds and images, computing credit scores, and detecting dangerous events, such as fraud, disease, and breaches of security.

Definition 3. Given n variables $x_1, \ldots, x_n$, where $A_i = \text{dom}(x_i)$, $i = 1, \ldots, n$, A **function of n-variables** over $x_1, \ldots, x_n$, denoted

$$f(x_1, \ldots, x_n),$$

and having codomain B, is a function $f : A_1 \times \cdots \times A_n \to B$ that assigns every n-tuple $(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n$ to some member $b \in B$. In function notation this is written as $f(a_1, \ldots, a_n) = b$.

Thus, a function of n variables is really just a function of a single variable, where the domain of that single variable is the set of all n-tuples of

$$A_1 \times \cdots \times A_n.$$

Example 18. The volume $v(r,h)$ of a cylinder is a function of two variables, namely the radius r of its circular cross section, and its height h, where the domain set of both r and h is $\mathcal{R}^+$, the set of positive real numbers. In this case the rule for $v(r,h)$ is $v(r,h) = \pi r^2 h$. For example, $v(2,3) = \pi 2^2 \cdot 3 = 12\pi$.

Although, technically speaking, a function f of n variables has a single domain set, namely $A_1 \times \cdots \times A_n$, we nevertheless refer to the sets $A_1, \ldots, A_n$ as the domain ses of f. Moreover, the **signature** of f is denoted $f(A_1, \ldots, A_n, B)$. For example, the signature of the volume function $v(r, h)$ in the previous example is $v(\mathcal{R}^+, \mathcal{R}^+, \mathcal{R}^+)$, where the first two $\mathcal{R}^+$'s represent the domains of r and h, while the third represents v's codomain.

An abridged version of a function's signature may be obtained by replacing sequences of repeating sets with a single occurrence of the set, followed by the number of times the set appears in the sequence. For example, the signature of $v(r, h)$ may be reduced to $v(\mathcal{R}^+, 3)$, since $\mathcal{R}^+$ appears three consecutive times in the unabridged signature.

Just as a function of a single variable is called a **unary function**, so a function of two variables is called a **binary function**. Moreover, every binary arithmetic, set, or logical operation may be considered a binary function. For example, binary addition has the signature $+(\mathcal{R}, 3)$, while binary OR has the signature $\vee(\mathcal{B}, 3)$. Although it is perfectly legal to write $+(3, 1) = 4$, we usually prefer to use + with *infix* notation, where the first input is written before +, and the second input is written after; such as $3+1 = 4$. A later lecture elaborates more on the ways that one can position a function's name in relation to its inputs.

Exercises

Note: set $\mathcal{R}$ denotes the set of real numbers, $\mathcal{I}$ denotes the set of integers, and $\mathcal{N}$ denotes the set of natural numbers $\{0, 1, \ldots\}$.

1. Why is $f(x) = 1/x$ not a function from $\mathcal{R}$ to $\mathcal{R}$?
2. Is $f(n) = \pm n$ a function from $\mathcal{I}$ to $\mathcal{I}$? Explain.
3. Let S denote the set of binary strings. For $s \in S$, let $f(s)$ denote the number of ones in s. Is f a function from S to $\mathcal{N}$? Explain.
4. Find the domain and range of the function that assigns to a bit string the number of bits in the string.
5. Find the domain and range of the function that assigns the number of left over bits after a bit string has been converted to bytes (i.e. blocks of 8 bits).
6. Find the domain and range of the function that assigns to a bit string the numerical position of the first one in the string, and that assigns the value of zero to the bit string consisting of all zeros.
7. Find the values of $\lceil 1.1 \rceil$, $\lceil -0.01 \rceil$, $\lceil 2.99 \rceil$, $\lceil \frac{1}{2} + \lfloor \frac{1}{2} \rfloor \rceil$.
8. Repeat the previous problem, but replace ceilings (floors) with floors (ceilings).
9. Let f map integers to integers. Is $f(n) = n^2 + 1$ one-to-one? How about $f(n) = \lceil n/2 \rceil$? Explain.
10. Repeat the previous problem, but replace "one-to-one", with "onto".
11. Let f map pairs of integers (m, n) to integers. Is $f(m, n) = |m| - |n|$ onto? What about $f(m, n) = m^2 - n^2$?
12. Give an example of a function from $\mathcal{N}$ to $\mathcal{N}$ that is i) one-to-one and onto, ii) one-to-one but not onto, iii) onto but not one-to-one, and iv) neither one-to-one, nor onto.
13. Determine whether each of the following are one-to-one correspondences from R to R.
    - **a.** $f(x) = 2x + 1$
    - **b.** $f(x) = x^2 + 1$
    - **c.** $f(x) = x^3$
    - **d.** $f(x) = (x^2 + 1)/(x^2 + 2)$
14. Show that $f(x) = x^3 + 1$ is a one-to-one correspondence. Assume that its domain and codomain are both equal to $\mathcal{R}$. Hint: show that every $y \in \mathcal{R}$ has a unique x that gets assigned to it.
15. Find algebraic rules for $f \circ g$ and $g \circ f$ if $f(x) = x^2 + 1$ and $g(x) = 2x + 3$.

16. Find algebraic rules for $f \circ g$ and $g \circ f$ if $f(x) = x^2$ and $g(x) = \lfloor x \rfloor$. Which is the larger of the two when x is positive?

17. If $(g \circ f)$ is one-to-one, give an example that shows that g does not necessarily have to be one-to-one.

18. If $(g \circ f)$ is onto, give an example that shows that f does not necessarily have to be onto.

19. For real x, find $f(S)$ when i) $f(x) = \lfloor x \rfloor$ and $S = \{-2, -1, 0, 1, 2\}$; and ii) $f(x) = x^2$ and $S = \{x | -3 \leq x \leq 2\}$.

20. For real x, find $f^{-1}(S)$ when i) $f(x) = \lfloor x \rfloor$ and $S = \{x | x > 4\}$; and ii) $f(x) = x^2$ and $S = \{x | 0 \leq x \leq 2\}$.

21. Let $f(n)$ be the function defined in Example 15. If $S = \{6, 12, 14, 45, 98\}$, then compute $f(S)$, the image of S under f.

22. Let $f(n)$ be the function defined in Example 15. Provide five members of the set $f^{-1}(\{2, 5\})$.

23. Let $g(n)$ be the function defined in Example 15. Explain why $g^{-1}(\{2, 5, 7\}) = \emptyset$.

24. Let $f(n)$ be the function defined in Example 15. Let T denote the power set of $\{1, 2, 3\}$. Provide a general description of the members of $f^{-1}(T)$.

25. Let f and g be two real-valued functions, where $f(x) = x^2 - 5x + 12$ and $g(x) = \sqrt{2x + 1}$. Then provide the following rules for $(f + g)(x)$, $(f - g)(x)$, $(fg)(x)$, and $(f/g)(x)$, and use them to compute $(f + g)(4)$, $(f - g)(4)$, $(fg)(4)$, and $(f/g)(4)$.

26. Let f and g be the functions defined in Example 15. Compute $(f \cup g)(30)$, $(f \cap g)(30)$, $(f \oplus g)(30)$, $(f - g)(30)$, and $\overline{f}(30)$ (assuming $\mathcal{U}$ is the set of all prime numbers between 1 and 40).

27. Let $A = \mathcal{N}$, and B denote the power set of $\mathcal{N}$. Then $f : A \to B$ is defined so that $f(n)$ is the set of all divisors of n; i.e. those positive integers that divide evenly into n. For example, $f(24) = \{1, 2, 3, 4, 6, 8, 12, 24\}$. Also, let $g : A \to B$ be defined so that $g(n)$ equals the set of all perfect squares that are less than or equal to n. For example, $g(24) = \{1, 4, 9, 16\}$. Compute $(f \cup g)(36)$, $(f \cap g)(36)$, $(f \oplus g)(36)$, $(f - g)(36)$, and $\overline{f}(36)$ (assuming the universe is all natural numbers $\mathcal{U}$ less than or equal to 40).

Exercise Solutions

1. $f(x) = 1/x$ not a function from R to R since $f(0)$ is undefined.

2. $f(n) = \pm n$ is not a function from I to I since f assigns to each n two outputs, rather than a single output, which is required for functions.

3. f a function from S to N since every bit string s is assigned to a unique natural number, that equals the number of zeros in s.

4. Domain: set of bit strings. Range: natural numbers $0, 1, 2, \ldots$.

5. Domain: set of bit strings. Range: all bit strings of length seven or less.

6. Domain: set of bit strings. Range: natural numbers $0, 1, 2, \ldots$.

7. $\lceil 1.1 \rceil = 2$, $\lceil -0.01 \rceil = 0$, $\lceil 2.99 \rceil = 3$, $\lceil \frac{1}{2} + \lfloor \frac{1}{2} \rfloor \rceil = 1$.

8. $\lfloor 1.1 \rfloor = 1$, $\lfloor -0.01 \rfloor = -1$, $\lfloor 2.99 \rfloor = 2$, $\lfloor \frac{1}{2} + \lceil \frac{1}{2} \rceil \rfloor = 1$.

9. $f(n) = n^2 + 1$ is not one-to-one since $f(-1) = f(1) = 2$. $f(n) = \lceil n/2 \rceil$ is not one-to-one since $f(1) = f(2) = 1$.

10. $f(n) = n^2+1$ is not onto since any negative integer has no preimage under f. But $f(n) = \lceil n/2 \rceil$ is onto since, for each integer m, $f(2m) = \lceil 2m/2 \rceil = m$.

11. $f(m, n) = |m| - |n|$ is onto since, for any integer $p \geq 0$, we have $f(p, 0) = |p| = p$, and for any integer $p < 0$ we have $f(0, p) = -|p| = p$. However, $f(m, n) = m^2 - n^2$ is not onto since 2 has not preimage. To see this, suppose

$$m^2 - n^2 = (m - n)(m + n) = 2$$

for some integers m and n. For example, suppose $m - n = 1$ and $m + n = 2$. Then $2m = 3$ which is has no integer solution. The same holds true for the system of equations $m - n = 2$ and $m + n = 1$.

12. i) $f(n) = n$ is one-to-one and onto, ii) $f(n) = 2n$ is one-to-one but not onto, iii) $f(n) = \lfloor n/2 \rfloor$ is onto but not one-to-one, and iv) $f(n) = 2\lfloor n/2 \rfloor$ is neither one-to-one, nor onto.

13. We have

   - **a.** $f(x) = 2x+1$ is a one-to-one correspondence, since for every real number y, $x = (y-1)/2$ is the unique real number for which $f(x) = y$.
   - **b.** $f(x) = x^2 + 1$ is not a one-to-one correspondence since $f(-1) = f(1) = 2$, and so f is not one-to-one.
   - **c.** $f(x) = x^3$ is a one-to-one correspondence since for every real number y, $x = \sqrt[3]{y}$ is the unique real number for which $f(x) = y$.
   - **d.** $f(x) = (x^2+1)/(x^2+2)$ is not a one-to-one correspondence since $f(-1) = f(1) = 2/3$, and so f is not one-to-one.

14. Given arbitrary real number y, the unique number that gets assigned to y is $\sqrt[3]{y-1}$ since $(\sqrt[3]{y-1})^3 + 1 = y$.

15. We have
$$(f \circ g)(x) = f(g(x)) = (2x+3)^2 + 1$$
and
$$(g \circ f)(x) = g(f(x)) = 2(x^2+1) + 3 = 2x^2 + 5.$$

16. We have
$$(f \circ g)(x) = f(g(x)) = (\lfloor x \rfloor)^2$$
and
$$(g \circ f)(x) = g(f(x)) = \lfloor x^2 \rfloor.$$
To see which grows faster, let $x = n + \epsilon$, where n is a nonnegative integer, and $0 < \epsilon < 1$. Then $(\lfloor x \rfloor)^2 = n^2$, while
$$\lfloor x^2 \rfloor = \lfloor (n+\epsilon)^2 \rfloor = \lfloor n^2 + 2n\epsilon + \epsilon^2 \rfloor > n^2$$
for sufficiently large n.

17. Let f and g both have domain and codomain equal to the set of natural numbers. Also, let $f(n) = 2n$ and $g(n) = \lfloor n/2 \rfloor$. Then g is not one-to-one, since, e.g., $g(2) = g(3) = 1$. However
$$(g \circ f)(n) = \lfloor (2n)/2 \rfloor = \lfloor n \rfloor = n$$
is a one-to-one correspondence, and is therefore one-to-one.

18. Let f and g both have domain and codomain equal to the set of natural numbers. Also, let $f(n) = 2n$ and $g(n) = \lfloor n/2 \rfloor$. Then f is not onto, since, e.g., no odd number is in f's range. However
$$(g \circ f)(n) = \lfloor (2n)/2 \rfloor = \lfloor n \rfloor = n$$
is a one-to-one correspondence, and is therefore onto.

19. In the first case we have $f(S) = S = \{-2, -1, 0, 1, 2\}$, since $\lfloor n \rfloor = n$ for every integer n. In the second case we have $f(S) = [0, 9]$, the interval of real numbers from 0 to 9.

20. For real x, find $f^{-1}(S)$ when i) $f(x) = \lfloor x \rfloor$ and $S = \{x | x > 4\}$; and ii) $f(x) = x^2$ and $S = \{x | 0 \leq x \leq 2\}$.

21. $f(6) = f(12) = \{2, 3\}$, $f(14) = f(98) = \{2, 7\}$, and $f(45) = \{3, 5\}$. Therefore,
$$f(\{6, 12, 14, 45, 98\}) = \{\{2, 3\}, \{2, 7\}, \{3, 5\}\}.$$

22. We need five positive integers, each of whose set of prime factors is equal to $\{2, 5\}$. Such integers include 10, 20, 40, 50, and 80.

23. Function $g(n)$ equals the set of prime numbers p for which $\sqrt{n} \leq p \leq n$. Thus for $g(n) = \{2, 5, 7\}$, we would need $\sqrt{n} \leq 2$ which means that $n \leq 4$. But then $7 \not\leq 4$, and so 7 could not be a member of $g(n)$. Therefore, no such n exists, meaning that $g^{-1}(\{2, 5, 7\}) = \emptyset$.

24. Let $A \subseteq \{1,2,3\}$ be a subset of $\{1,2,3\}$. Notice that $f^{-1}(A) = \emptyset$ if $1 \in A$. This is because 1 is not a prime number, and so $f(n) \neq A$, for any A for which $1 \in A$. Thus, this only leaves the subsets $\{2\}$, $\{3\}$, and $\{2,3\}$. Moreover, $f^{-1}(\{2\})$ equals all positive powers of 2, $f^{-1}(\{3\})$ equals all positive powers of 3, and $f^{-1}(\{2,3\})$ equals all numbers of the form $2^i 3^j$, where $i, j \geq 1$. Putting all this together, we have $f^{-1}(T)$ is the set of all numbers of the form $2^i 3^j$, where $i, j \geq 0$, and $i + j \geq 1$. Such numbers include, 2,3,4,6,8,9,12,24, etc..

25. We have $(f+g)(x) = x^2 - 5x + 12 + \sqrt{2x+1}$, $(f-g)(x) = x^2 - 5x + 12 - \sqrt{2x+1}$, $(fg)(x) = (x^2 - 5x + 12)\sqrt{2x+1}$, and $(f/g)(x) = (x^2 - 5x + 12)/\sqrt{2x+1}$. Moreover, $(f+g)(4) = f(4) + g(4) = 11$, $(f-g)(4) = 5$, $(fg)(4) = 24$, and $(f/g)(4) = 8/3$.

26. We have $f(30) = \{2,3,5\}$, $g(30) = \{7,11,13,17,19,23,29\}$. Also

$$(f \cup g)(30) = f(30) \cup g(30) = \{2,3,5,7,11,13,17,19,23,29\},$$

$$(f \cap g)(30) = f(30) \cap g(30) = \emptyset$$

$$(f \oplus g)(30) = f(30) \oplus g(30) = \{2,3,5,7,11,13,17,19,23,29\},$$

$$(f - g)(30) = f(30) - g(30) = \{2,3,5\},$$

and

$$\overline{f}(30) = \{1,2,\ldots,40\} - \{2,3,5\}.$$

27. We have $f(36) = \{1,2,3,4,6,9,12,18,36\}$, $g(36) = \{1,4,9,16,25,36\}$. Also

$$(f \cup g)(36) = f(36) \cup g(36) = \{1,2,3,4,6,9,12,16,18,25,36\},$$

$$(f \cap g)(36) = f(36) \cap g(36) = \{1,4,9,36\},$$

$$(f \oplus g)(36) = f(36) \oplus g(36) = \{2,3,6,12,16,18\},$$

$$(f - g)(36) = f(36) - g(36) = \{2,3,6,12,18\},$$

and

$$\overline{f}(36) = \{1,2,\ldots,40\} - \{1,2,3,4,6,9,12,18,36\} = \{5,7,8,10,\ldots,17,19,\ldots,35,\ldots,37,40\}.$$