

Introduction to Graphs

Introduction

Graphs represent a fundamental structure that is used extensively in computing. We often hear about *cyber space* as being a new frontier for mankind, and if we look at the structure of cyberspace, we see that it is structured as a graph; in other words, it consists of computers, and connections between them. Some applications of graphs include

- representing electronic circuits
- models for object interactions (e.g. used in the Unified Modeling Language)
- showing ordering relationships between a set of tasks
- modeling networks and network traffic

Definitions

A **Graph** is a pair of sets V, E , where

Graph $G = (V, E)$ V is a set of **vertices**, also called **nodes**, while E is a set whose members are pairs of vertices and are called **edges**. Each edge may be written as a tuple of the form (u, v) , where $u, v \in V$.

Loop Edge $e = (u, u)$ is called a **loop** since it connects vertex u to itself.

Adjacency and Incidence If $e = (u, v)$ is an edge, then we say that u is **adjacent** to v , and that e is **incident** with u and v .

Order $|V| = n$ is called the **order** of G .

Size $|E| = m$ is called the **size** of G .

Path A **path** P of length k in a graph is a sequence of vertices $P = v_0, v_1, \dots, v_k$, such that $(v_i, v_{i+1}) \in E$ for every $0 \leq i \leq k - 1$.

Simple Path $P = v_0, v_1, \dots, v_k$ and the vertices v_0, v_1, \dots, v_k are all distinct.

Cycle A **cycle** is a path that begins and ends at the same vertex.

Geometrical Representation obtained by representing each vertex as a figure (usually a circle) on a two-dimensional plane, and each edge $e = (u, v)$ as a smooth arcs that connects vertex u with vertex v .

Degree The **degree** of a vertex v , denoted as $\deg(v)$, equals the number of edges that are incident with v . Note: loop edges are counted twice.

Handshaking Theorem: $\sum_{v \in V} \deg(v) = 2|E|$.

Proof of the Handshaking Theorem. Every edge adds one to the degree of exactly 2 vertices. Hence, in summing the degrees one gets a 2 to 1 ratio between total degree and edges, which is exactly what the Handshaking theorem states.

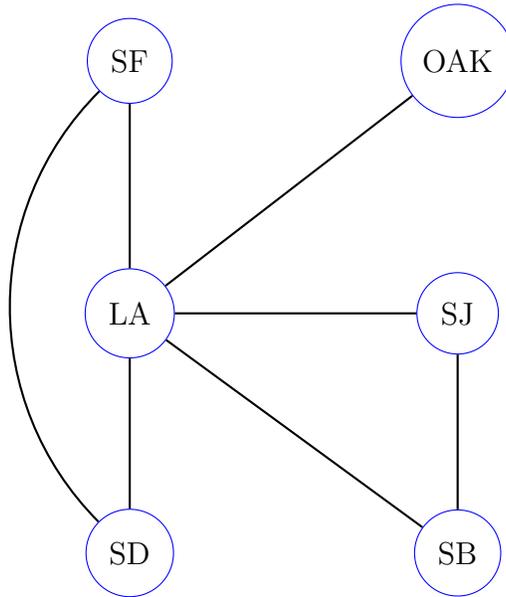


Figure 1: Graphical Representation of G from Example 1

Example 1. Let $G = (V, E)$, where

$$V = \{SD, SB, SF, LA, SJ, OAK\}$$

are cities in California, and

$$E = \{(SD, LA), (SD, SF), (LA, SB), (LA, SF), (LA, SJ), (LA, OAK), (SB, SJ)\}$$

are edges, each of which represents the existence of one or more flights between two cities. Figure 1 shows a graphical representation of G . G has order 6 and size 7.

Figure 2 shows a simple path of length 4. Figure 3 shows a cycle of length 3. Let's verify the Handshaking theorem.

$$\begin{aligned} \deg(SF) + \deg(LA) + \deg(SD) + \deg(OAK) + \deg(SJ) + \deg(SB) = \\ 2 + 5 + 2 + 1 + 2 + 2 = 14 = 2 \cdot 7 = 2|E|. \end{aligned}$$

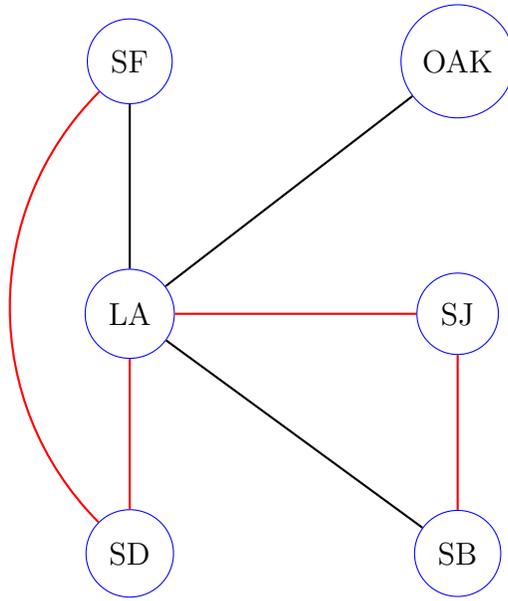


Figure 2: Simple path (in red) $P = SF, SD, LA, SJ, SB$ of length 4

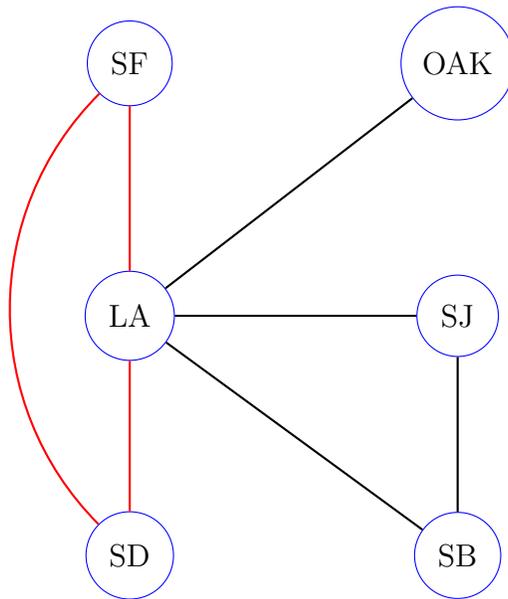


Figure 3: Cycle (in red) $C = SF, SD, LA, SF$ of length 3

Directed Graphs

A **directed graph** is a graph $G = (V, E)$ for which each edge represents an *ordered* pair of vertices.

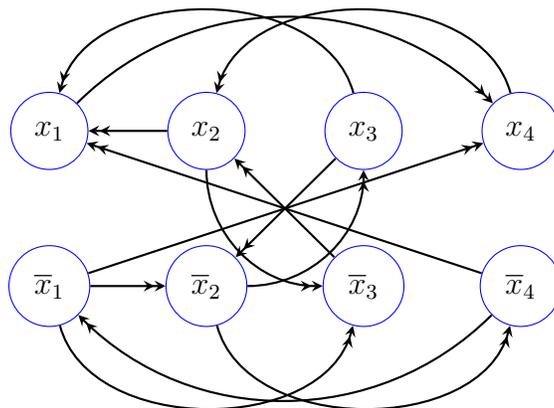
- If $e = (u, v)$ is an edge of a directed graph, then u is called the **start vertex** of the edge, while v is called the **end vertex** of the edge.
- the **in-degree** of vertex v , denoted $deg^+(v)$, is the number of edges for which v is the end vertex.
- the **out-degree** of vertex v , denoted $deg^-(v)$, is the number of edges for which v is the start vertex.
- similar to the handshaking theorem

$$\sum_{v \in V} deg^+(v) = \sum_{v \in V} deg^-(v) = |E|.$$

Example 2. The following directed graph is an example of an **implication graph**. Such graphs can be used to solve logic problems for which each logical formula is a binary disjunction. This graph represents the set of binary disjunctions

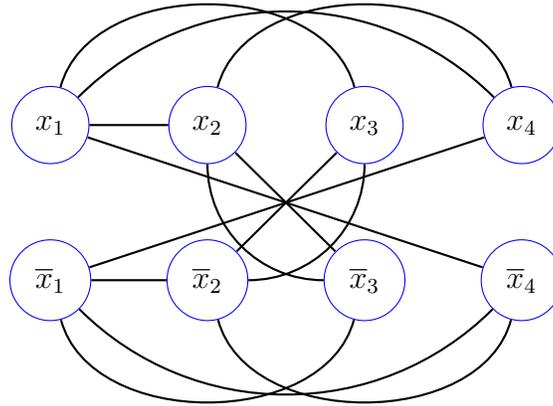
$$\mathcal{C} = \{(\bar{x}_2, \bar{x}_3), (x_2, \bar{x}_4), (x_1, \bar{x}_3), (x_2, x_3), (x_1, x_4), (\bar{x}_1, x_4), (x_1, \bar{x}_2)\}.$$

For example, (\bar{x}_2, \bar{x}_3) denotes the binary disjunction $(\neg x_2 \vee \neg x_3)$.



Example 3. For the graph in Example 2, verify the Handshaking theorem for directed graphs.

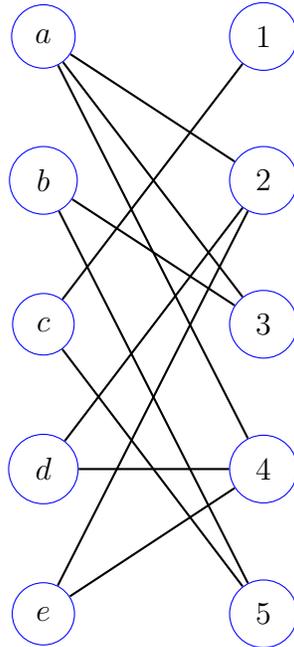
Given a directed graph $G = (V, E)$, the **underlying undirected graph (UUG)** of G , denoted $\text{UUG}(G)$, is the undirected graph that has the same vertex set and edge set as G , but now each edge is viewed as an *unordered* pair of vertices. of a directed graph $G = (V, E)$ is the undirected graph that is obtained by making the edges of G undirected. For example, below is the UUG of the graph from Example 2. Notice that vertices x_2 and \bar{x}_3 now have two edges connecting them. When this occurs we call the graph a **multigraph**.



Bipartite Graphs

A bipartite graph is a graph $G = (V, E)$ whose vertex set V can be *partitioned* into two sets V_1 and V_2 so that every edge $e \in E$ is incident with one vertex in V_1 and one vertex in V_2 .

Example 4. The following is an example of a bipartite graph.



Theorem 2. A graph is bipartite if and only if it does not contain an odd cycle.

Proof of Theorem 2. First suppose G is bipartite. Notice that any cycle in G must make an equal number of left-to-right and right-to-left edge traversals, since the path ends on the same side (either left or right) that it begins. Therefore the number of edge traversals (i.e. cycle length) must be even.

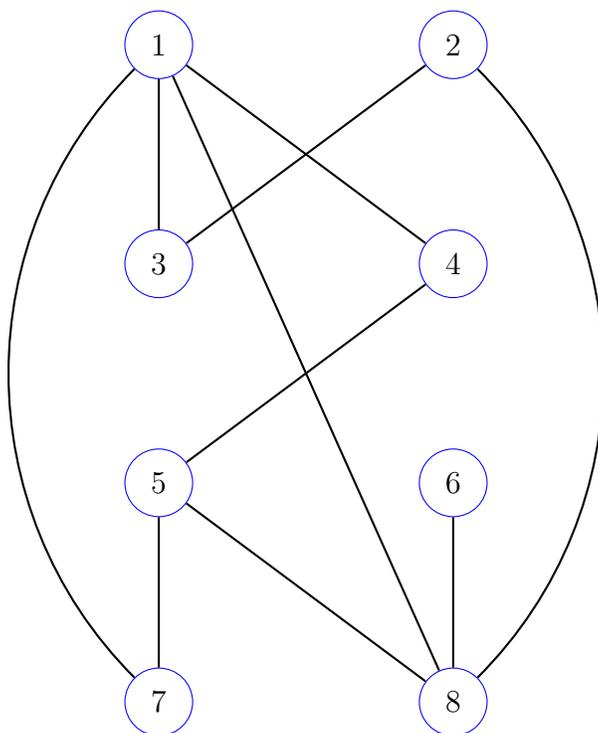
Now suppose G has no odd cycles. Then we may partition the vertices of G into disjoint sets S_1, S_2, \dots, S_k , i.e.

$$S_1 \cup S_2 \cup \dots \cup S_k = V,$$

and $S_i \cap S_j = \emptyset$ when $i \neq j$. To begin, let $S_1 = \{v\}$ for some vertex $v \in V$ and color v red. Thus, S_1 is called a *red* set. Next, let $S_2 = \{w \in V \mid (v, w) \in E\}$ be the set of all vertices that are adjacent to v (we assume G has no loops). Color all these vertices blue. Thus, S_2 is called a *blue* set. Notice that no two blue vertices in S_2 can be adjacent to each other, since this would create an odd cycle with v . Now suppose sets S_1, S_2, \dots, S_j have been defined, and that they alternate between red and blue. Let S_{j+1} be the set of all the vertices that are adjacent to a vertex in S_j , and have yet to be colored (i.e. added to a previous set). Assign these vertices a color (either red or blue) that is different from the color of S_j .

To finish the proof, notice that, for any two vertices $u, w \in S_{j+1}$ there is a path from v to u of length j , and a path from v to w of length j . Thus, $(u, w) \notin E$, since this would create an odd cycle of length $2j + 1$. Finally, notice that there cannot be two blue vertices u and w for which $(u, w) \in E$, and u and w are in different blue sets, say $u \in S_p$ and $w \in S_q$, with $p + 1 < q$. This is not possible since the edge (u, w) would have placed w in S_{p+1} and would have colored it red. Hence, no two blue vertices can be adjacent, and the same is true for any two red vertices via a similar argument. Hence, G is bipartite with the partition of vertices consisting of the red vertices in one set, and the blue vertices in the other. \square

Example 5. Use the proof technique in Theorem 2 to determine if the following graph is bipartite.



Graph Connectivity

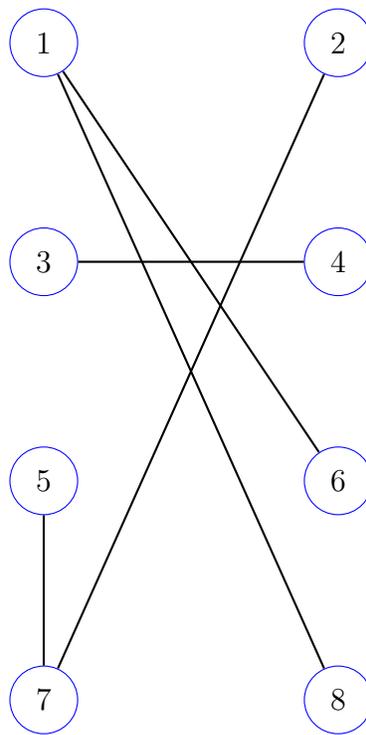
A graph $G = (V, E)$ is said to be **connected** iff for any two vertices $u, v \in V$ there is a path P from u to v . Moreover, if G is directed then the following terms apply.

Strongly Connected for every $u, v \in V$, there is a *directed path* from u to v ; i.e. every step from one vertex to an adjacent vertex must follow the direction of the edge (from start vertex to end vertex).

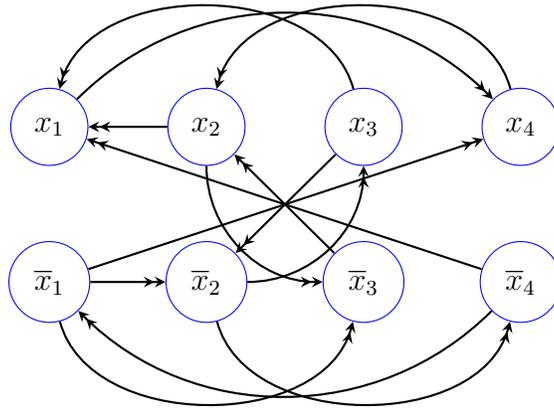
Weakly Connected the underlying undirected graph of G is connected.

A **subgraph** of a graph G is a graph that uses some of the edges and vertices of G . For example, a path in a graph G is a subgraph whose vertices are the vertices visited along the path, and whose edges are the edges traversed along the path. Given a graph $G = (V, E)$, a **connected component** of G is a subgraph H of G that is connected, and for which no other subgraph of G that contains H is connected. In other words, if one adds another vertex of G to H , then the resulting subgraph is no longer connected.

Example 6. Determine the connected components for the following graph.



Example 7. Is the following graph G strongly connected? Weakly connected? Provide the strongly-connected components of G .



Directed Acyclic Graphs

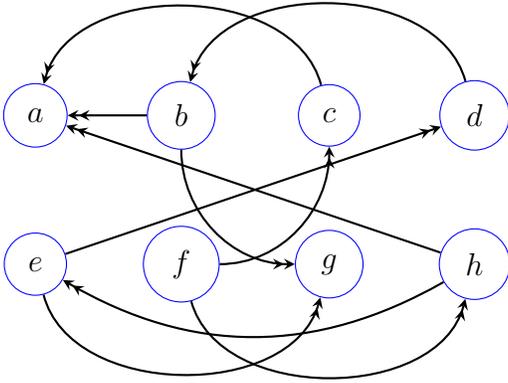
A graph is said to be **acyclic** iff it does not have any cycles. Directed acyclic graphs (DAGS) have several applications in scheduling, planning, and modeling dependencies between entities.

Theorem 3. If a directed graph is acyclic, then it has at least one vertex with an out-degree equal to zero, and at least one vertex with an in-degree equal to zero.

Proof of Theorem 3. Suppose by way of contradiction that every vertex of a DAG $G = (V, E)$ has an out-degree of at least one. Let $P = v_1, v_2, \dots, v_k$ be a longest simple path in G . Then, since v_k has positive out-degree, there is another vertex u for which (v_k, u) is an edge. Now if $u \notin P$, then $P = v_1, v_2, \dots, v_k, u$ is a longer simple path, a contradiction. Therefore, $u \in P$ which means G has a cycle (why?), a contradiction. \square

A **topological sort** for a DAG $G = (V, E)$ is an ordering of V so that, if $(u, v) \in E$, then u comes before v in the ordering.

Example 8. Provide a topological sort for the following DAG.



Additional Properties of Graphs

Simple G is **simple** if it has no loops, and is not a multigraph.

Complete G is complete iff it is simple and every pair of vertices in G are adjacent.

Clique A **clique** in a graph G is a subset of vertices G for which each pair is adjacent. In other words, a clique in a graph G is a complete subgraph of G . A **k-clique** is a clique having cardinality equal to k .

Independent Set An **independent set** in a graph G is a subset of vertices in G , no two of which are adjacent.

Vertex Cover A **vertex cover** in a graph $G = (V, E)$ is a subset of vertices for which every edge $e \in E$ is incident with at least one vertex in the set.

Hamilton Path A **Hamilton path** in a graph G is a simple path that visits every vertex of G . A **Hamilton Cycle** in G is a Hamilton path with one additional edge that is incident with the first and last vertices in the path.

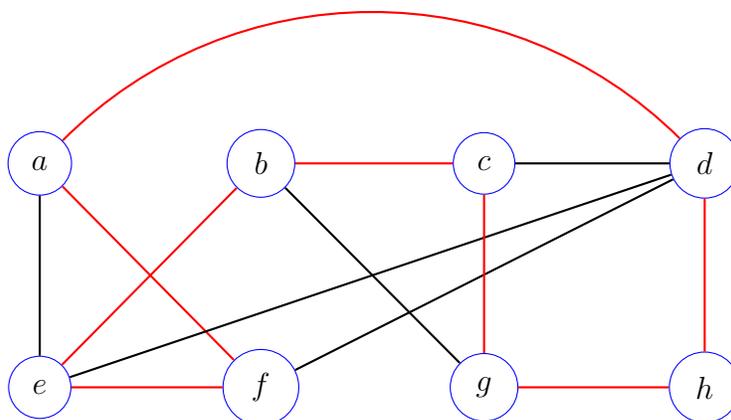
Example 9. The following Graph has a Hamilton Cycle (shown in red)

$$C = a, d, h, g, c, b, e, f, a$$

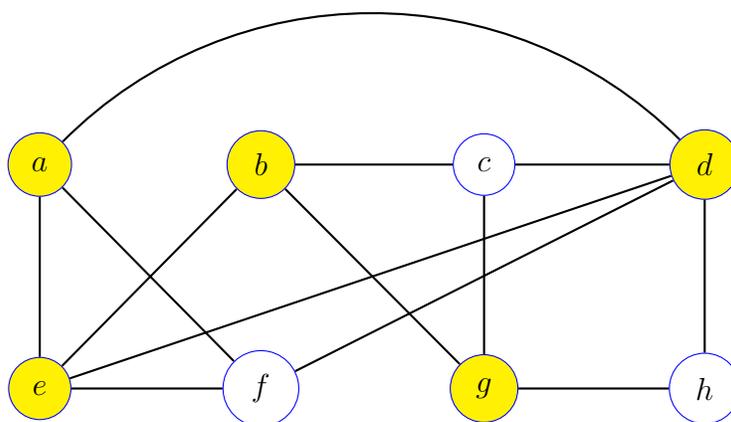
which means that

$$P = a, d, h, g, c, b, e, f$$

is a Hamilton path from a to f . It also has several 3-cliques. For example, $\{a, e, f\}$ is one such clique. However, it has no 4-cliques (verify). Also, $\{a, c, h\}$ is an independent set in G .



The following is the same graph but with yellow vertices $\{a, b, d, e, g\}$ forming a vertex cover for G .



Graph Isomorphism

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then we say that G_1 is **isomorphic** to G_2 if and only there exists a one-to-one correspondence $f : V_1 \rightarrow V_2$ for which

$$(a, b) \in E_1 \iff (f(a), f(b)) \in E_2.$$

- If G_1 is isomorphic to G_2 , we write $G_1 \cong G_2$.
- Graph isomorphism has practical importance because often times when we analyze system A and model it with a graph G_1 , we may notice that it is structurally equivalent to graph G_2 which was used to model system B . Thus in some sense these two systems represent the same entity from a structural point of view.
- The problem of determining if two graphs are isomorphic represents an important problem in theoretical computer science, in that no efficient algorithm has been devised, yet no one has been able to prove that it is impossible to devise such an algorithm.
- A **graph-theoretic property** is any property of a graph G that holds for any other graph G' that is isomorphic to G .

Theorem 4. If $G \cong G'$ then both graphs

1. have the same degree sequence.
2. have the same number of connected components.
3. have the same order and size.
4. have the same cycle composition.

In other words, degree sequence, connectivity, order, size, and cycle composition are all graph-theoretic properties. Other graph-theoretic properties include maximum clique size, maximum independent-set size, minimum cover size, and whether or not a graph possesses a Hamilton path and/or Hamilton cycle.

Example 10. Are the following graphs isomorphic?

$$G_1 = (\{1, 2, 3, 4\}, \{(1, 2), (2, 3), (3, 4), (1, 4)\}),$$

and

$$G_2 = (\{a, b, c, d\}, \{(a, c), (a, d), (b, c), (b, d)\}).$$

If yes, provide the one-to-one correspondence between vertices, and the corresponding one-to-one correspondence between edges. If no, explain why.

Example 11. Are the following graphs isomorphic?

$$G_1 = (\{1, 2, 3, 4, 5\}, \{(1, 2), (2, 3), (2, 4), (2, 5), (3, 5), (4, 5)\}),$$

and

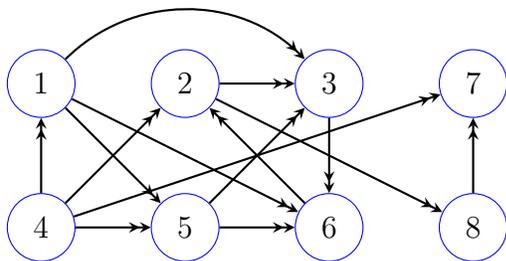
$$G_2 = (\{a, b, c, d, e\}, \{(a, b), (a, c), (b, d), (b, c), (c, d), (d, e)\}).$$

If yes, provide the one-to-one correspondence between vertices, and the corresponding one-to-one correspondence between edges. If no, explain why.

Exercises

- Let $G = (V, E)$ be an undirected graph, where $V = \{1, 2, \dots, 8\}$, and

$$E = \{(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 6), (2, 8), (3, 5), (3, 6), (4, 5), (4, 7), (5, 6), (7, 8)\}.$$
 - Provide a graphical representation of G .
 - Determine the order and size of G .
 - Provide a simple path P that has the greatest length among all simple paths.
 - Provide a cycle C that has the greatest length among all cycles.
- Verify the Handshaking Theorem for the graph from Exercise 1.
- For the following directed graph, verify the Handshaking theorem by i) summing all the in-degrees, and ii) summing all the out-degrees.



- Draw the implication graph for the following set of CNF clauses.

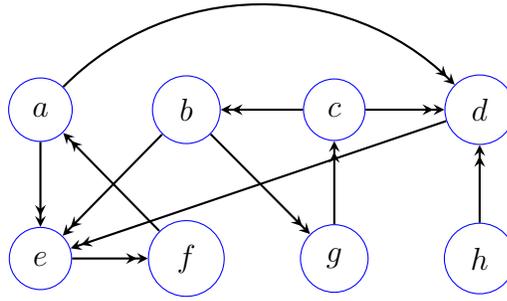
$$(\bar{x}_2, \bar{x}_3), (x_2, \bar{x}_4), (x_1, \bar{x}_3), (x_2, x_3), (x_1, x_4), (\bar{x}_1, x_4), (x_1, \bar{x}_2).$$

Hint: (\bar{x}_2, \bar{x}_3) stands for $\neg x_2 \vee \neg x_3$. Convert this formula to two different conditional formulas.

- Show that the following graph is bipartite by determining two sets V_1 and V_2 for which $V_1 \cup V_2 = V$ and $e = (u, v) \in E$ iff $u \in V_1$ and $v \in V_2$. Draw the graph in a way that draws the V_1 vertices on the left and the V_2 vertices on the right. $G = (V, E)$, where $V = \{1, 2, \dots, 10\}$, and

$$E = \{(1, 5), (1, 9), (1, 2), (2, 3), (2, 7), (2, 8), (3, 4), (3, 10), (4, 6), (4, 7), (5, 6), (5, 7), (6, 10), (7, 9), (8, 10)\}.$$
- For the graph in the previous exercise, determine its largest independent set, and a vertex cover of least size.
- Explain why a bipartite graph having at least one edge has a 2-clique, but no k -cliques, for all $k \geq 3$.
- Provide the connected components for the simple graph $G = (V, E)$, where $V = \{1, \dots, 7\}$ and

$$E = \{(1, 3), (1, 5), (2, 4), (3, 5), (3, 7)\}.$$
- Draw the strongly connected components of the following graph. Hint: there are three of them.

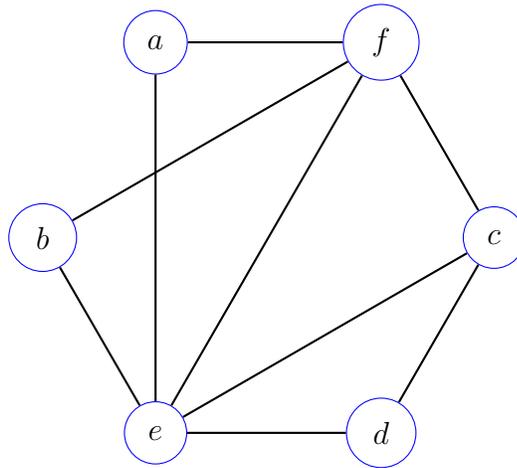


10. Draw the strongly connected components of the implication graph from Exercise 4.
11. Determine a topological sort for the directed graph having vertex set a-k. and edges

$$\{(j, a), (j, g), (a, b), (a, e), (b, c), (c, k), (d, e), (e, c), (e, f), (e, i), (f, k), \\ (g, d), (g, e), (g, h), (h, e), (h, i), (i, f), (i, k)\}.$$

Draw the graph vertices in a linear fashion so that all edges are oriented from left to right.

12. For the following graph shown below, i) find a clique of largest size, ii) find an independent set of largest size, iii) find a vertex cover of smallest size. Does this graph have a Hamilton path? Hamilton cycle?



13. Draw the complete graphs K_5 and K_6 . In general, how many edges does K_n have?
14. The **cubic graph** Q_n is a simple graph that has 2^n vertices, with each vertex being labeled with a binary string of length n . Furthermore, two vertices are adjacent iff they differ by at most one bit. For example, if $n = 4$, the vertex 1001 is adjacent to 1011, since they only differ in the third bit place. Draw Q_1, Q_2 , and Q_3 . In general, what is the order and size of Q_n ?
15. Use mathematical induction to prove that Q_n has a Hamilton cycle when $n \geq 2$. Hint: explain why Q_n may be viewed as consisting of two copies of Q_{n-1} , with corresponding vertices of each copy connected by an edge.
16. Prove that Q_n is bipartite and that its two partition sets V_1 and V_2 have equal size.

17. Are the following graphs isomorphic?

$$G_1 = (\{1, 2, 3, 4, 5\}, \{(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 5)\}),$$

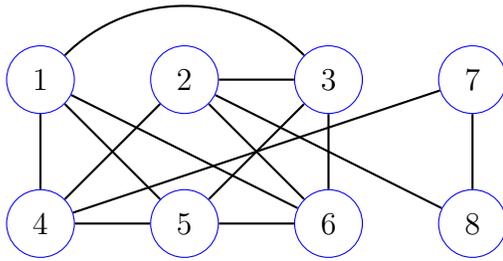
and

$$G_2 = (\{a, b, c, d, e\}, \{(a, d), (a, e), (e, d), (b, c), (a, b), (b, d), (c, e)\}).$$

If yes, provide the one-to-one correspondence between vertices, and the corresponding one-to-one correspondence between edges. If no, explain why.

Exercise Solutions

1. We have the following.



a.

b. G has order 8 and size 14.

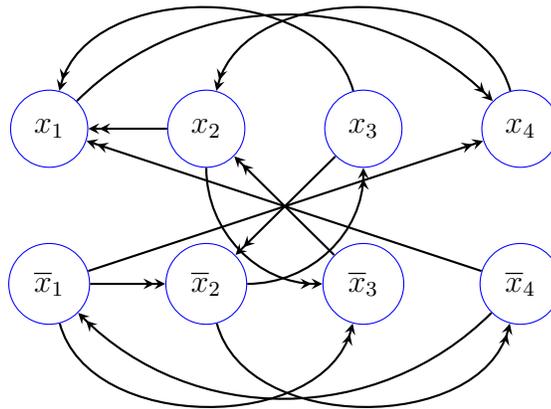
c. $P = 1, 3, 2, 6, 5, 4, 7, 8$

d. $C = 1, 5, 6, 3, 2, 8, 7, 4, 1$

2.

3.

4. The implication graph G_C is shown below.



5.

6.

7.

8.

9.

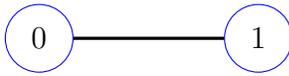
10.

11.

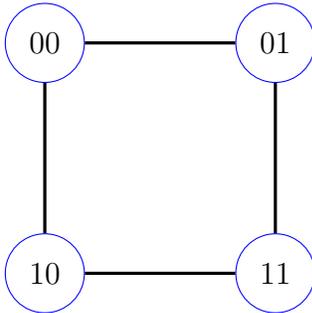
12.

13.

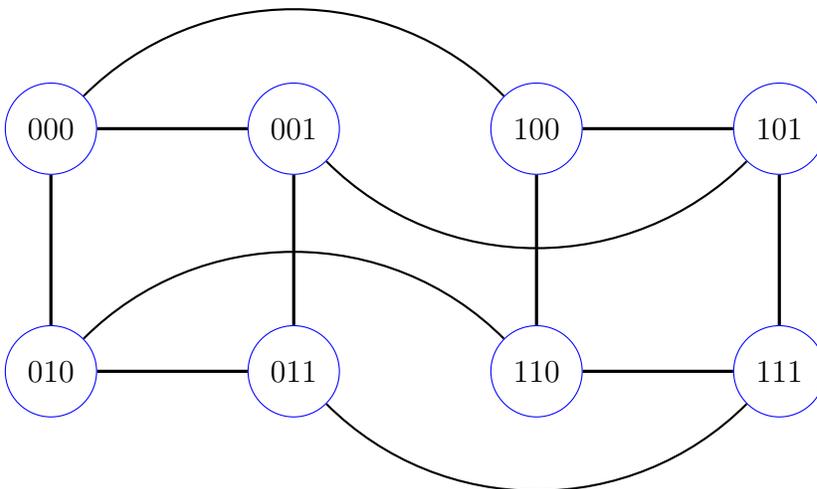
14. Q_1 is shown below.



Q_2 is shown below.



Q_3 is shown below.



In general, Q_n has order 2^n and size $n2^{n-1}$. The latter is computed using the Handshaking theorem, and the fact that all vertices have degree equal to n .

15.

16.

17.