Mathematical Induction

Weak Mathematical Induction

Goal prove that statement $P(n)$ is true $\forall n \in \mathcal{N}$

Basis Step prove that $P(0)$ is true

Induction Step assume there exists an n such that $P(n)$ is true. Show that $P(n+1)$ is true. Note that the statement "there exists an n such that $P(n)$ is true" is called the **inductive assumption**.

A proof by induction is analogous to knocking over a row of dominoes by pushing over the first domino (basis step) in the row, and the observation that, if domino n falls, then so will domino $n+1$ (inductive step).

Example 1. Use mathematical induction to prove that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Example 2. Use mathematical induction to prove that $n! > 2^n$ for $n \geq 4$

Example 3. Use mathematical induction to prove that n^2 is odd if n is odd.

Example 4. Prove the generalized form of De Morgan's Rule: namely

$$\neg(p_1 \vee p_2 \vee \cdots \vee p_n) \Longleftrightarrow (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n).$$

Example 5. Use mathematical induction to prove that any number $n \geq 13$ can be written as $n = 4k_1 + 5k_2$, where $k_1, k_2 \geq 0$ are integers. For example, $13 = 4(2) + 5(1)$.

Strong Induction

Strong induction uses a stronger inductive assumption. The inductive assumption "Assume $P(n)$ is true for some $n \geq 0$" is replaced by "Assume $P(k)$ is true *for every* $0 \leq k \leq n$".

Example 6. A game consists of two players and two piles of matches, each having the same number $n \geq 1$ of matches. The players then take turns, where each turn allows one player to remove any number of matches from one of the piles. The game continues until all the matches have been removed from both piles. The winner is the player who removes the last match(es). Use strong mathematical induction to prove that the player who goes second can always win.

Proposition. Let a be a nonnegative integer and b be a positive integer. Let r be the remainder of dividing a by b, i.e. $r = a \bmod b$. Then $\gcd(a, b) = \gcd(b, r)$.

Proof of Proposition. Since $r = a \bmod b$, there is a q for which

$$a = bq + r.$$

Let D_{ab} denote the set of common divisors of a and b, and D_{br} denote the set of common divisors of b and r. We show that $D_{ab} = D_{br}$. First consider $d \in D_{ab}$. Then we have $dk_1 = a$ and $dk_2 = b$ for some integers k_1 and k_2. But then

$$r = a - bq = dk_1 - dk_2q = d(k_1 - k_2q),$$

which implies d divides r and $d \in D_{br}$. Now suppose $d \in D_{br}$. Similarly, we have $dk_1 = b$ and $dk_2 = r$, which implies d divides a since

$$a = dk_1q + dk_2 = d(k_1q + k_2).$$

□

Example 7. Use the previous proposition and strong mathematical induction to prove or every two nonnegative integers a and b, there exist integers s and t such that

$$\gcd(a, b) = sa + tb.$$

Example 8. Use strong mathematical induction to prove that every positive integer $n \geq 2$ can be written as the product of one or more prime numbers.

Example 9. Use strong mathematical induction to prove that every nonnegative integer can be written as the sum of powers of two (i.e. every nonnegative integer has a binary representation).

Exercises

1. Use mathematical induction to prove that $1 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

2. Use mathematical induction to prove that $1 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.

3. Use mathematical induction to prove that, for all integers $k \geq 1$ and arbitrary $\epsilon > 0$,
$$\lim_{n\to\infty} \frac{\log^k n}{n^\epsilon} = 0.$$
Hint: use L'Hospital's rule that states, if $f(n)$ and $g(n)$ are differentiable functions that increase to infinity, then
$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}.$$

4. Use mathematical induction to prove that n^2 is even if n is even.

5. Use mathematical induction to prove that any number $n \geq 8$ can be written as $n = 3k_1 + 5k_2$, where $k_1, k_2 \geq 0$ are integers. In other words, every $n \geq 8$ may be written as a **nonnegative linear combination** of 3's and 5's. We use the term *nonnegative* since $k_1, k_2 \geq 0$.

6. Use mathematical induction to prove that any number $n \geq 18$ can be written as $n = 4k_1 + 7k_2$, where $k_1, k_2 \geq 0$ are integers.

7. Determine which numbers n can be written as $n = 4k_1 + 11k_2$, where $k_1, k_2 \geq 0$ are integers. Prove your statement using mathematical induction.

8. A jigsaw puzzle is put together by successively joining pieces that fit together into blocks. A move is made each time a piece is added to a block, or when two blocks are joined. Use strong mathematical induction to prove that, no matter what sequence of moves, exactly $n - 1$ moves is required to assemble a puzzle having n pieces.

9. Consider the following variation of the game called **Nim**. The game begins with $n \geq 1$ matches. Two players take turns removing matches, either one, two, or three at a time. The player removing the last match loses. Use strong mathematical induction to prove that, assuming both players use optimal strategies, the second player can only win when $n \bmod 4 = 1$. Otherwise, the first player will win.

10. Use strong induction to prove that $\sqrt{2}$ is irrational. In particular, show that $\sqrt{2} \neq n/b$ for any $n \geq 1$ and fixed integer $b \geq 1$.

Exercise Solutions

1. **Basis step** $1 = \frac{(1)(2)(3)}{6}$.

   Inductive step Assume $1^2 + 2^2 + \dots n^2 = \frac{n(n+1)(2n+1)}{6}$ for some $n \geq 1$. Show:

   $$1^2 + 2^2 + \dots n^2 + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}.$$

   $$1^2 + 2^2 + \dots n^2 + (n+1)^2 = \frac{n(n+1)(2n+1}{6} + (n+1)^2 =$$
   $$\frac{(n+1)}{6}(n(2n+1) + 6(n+1)) = \frac{(n+1)}{6}(2n^2 + 7n + 6) = \frac{(n+1)}{6}(n+2)(2n+3) =$$
   $$\frac{(n+1)(n+2)(2n+3)}{6},$$

   where the first equality is due to the inductive assumption.

2. **Basis step** $1 = \left(\frac{(1)(2)}{2}\right)^2$.

   Inductive step Assume $1^3 + 2^3 + \dots n^3 = \left(\frac{n(n+1)}{2}\right)^2$ for some $n \geq 1$. Show:

   $$1^3 + 2^3 + \dots n^3 + (n+1)^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2.$$

   $$1^3 + 2^3 + \dots n^3 + (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 =$$
   $$(n+1)^2\frac{n^2 + 4(n+1)}{4} = (n+1)^2\left(\frac{(n+2)}{2}\right)^2 = \left(\frac{(n+1)(n+2)}{2}\right)^2,$$

   where the first equality is due to the inductive assumption.

3. For simplicity, we assume that $\log n$ has base e, so that $(\log n)' = \frac{1}{n}$.

   Basis step We have by the above derivative and the power rule of differentiation,

   $$\lim_{n\to\infty} \frac{\log^1 n}{n^\epsilon} = \lim_{n\to\infty} \frac{(\log n)'}{(n^\epsilon)'} =$$
   $$\lim_{n\to\infty} \frac{1}{n\epsilon n^{\epsilon-1}} = \frac{1}{\epsilon}\lim_{n\to\infty} \frac{1}{n^\epsilon} = 0,$$

   since the numerator is fixed at 1, and the denominator increases to infinity.

   Inductive step Assume $\lim_{n\to\infty} \frac{\log^k(n)}{n^\epsilon} = 0$ for some $k \geq 1$. Show:

   $$\lim_{n\to\infty} \frac{\log^{k+1}(n)}{n^\epsilon} = 0.$$

   Then by L'Hospital's Rule, the chain rule, and the power rule of differentiation,

   $$\lim_{n\to\infty} \frac{\log^{k+1}(n)}{n^\epsilon} = \lim_{n\to\infty} \frac{(\log^{k+1}(n))'}{(n^\epsilon)'} =$$

$$\lim_{n\to\infty} \frac{(k+1)\log^k(n)}{n\epsilon \cdot n^{\epsilon-1}} = \frac{(k+1)}{\epsilon} \lim_{n\to\infty} \frac{\log^k(n)}{n^\epsilon} = 0,$$

where the last equality is due to the inductive assumption.

4. **Basis step** $0^2 = 0$ is even.

   Inductive step Assume n and n^2 are both even for some $n \geq 0$.
   Show: $(n+2)^2$ is even (since $n+2$ is the next even number). Since n^2 is even, there is an integer k for which $n^2 = 2k$. Then

$$(n+2)^2 = n^2 + 4n + 4 = 2k + 4n + 4 = 2(k + 2n + 2)$$

   is even since it has a factor of 2.

5. Prove that $n \geq 8$ can be written as $n = 3k_1 + 5k_2$, where $k_1, k_2 \geq 0$ are integers.

   Basis step $n = 8 = 3(1) + 5(1)$.

   Inductive step Assume n is a nonnegative linear combination of 3's and 5's, for some $n \geq 8$.
   Show: $(n+1)$ is a linear combination of 3's and 5's. Since n is a nonnegative linear combination of 3's and 5's, we may write it as $n = 3k_1 + 5k_2$, with $k_1, k_2 \geq 0$. Then

$$n + 1 = 3k_1 + 5k_2 + 3(2) - 5 = 3(k_1 + 2) + 5(k_2 - 1).$$

   Case 1. $k_2 \geq 1$. Then the above nonnegative linear combination suffices. **Case 2.** $k_2 = 0$. Then $n = 3k_1$, where $k_1 \geq 3$. Thus,

$$n + 1 = 3(k_1 - 3) + 5(2)$$

   is a nonnegative linear combination of 3's and 5's. □

6. Prove that if $n \geq 18$ can be written as $n = 4k_1 + 7k_2$, where $k_1, k_2 \geq 0$ are integers.

   Basis step $n = 18 = 4(1) + 7(2)$.

   Inductive step Assume n is a nonnegative linear combination of 4's and 7's, for some $n \geq 18$.
   Show: $(n+1)$ is also a nonnegative linear combination of 4's and 7's. Since n is a nonnegative linear combination of 4's and 7's, we may write it as $n = 4k_1 + 7k_2$, for $k_1, k_2 \geq 0$. Then

$$n + 1 = 4k_1 + 7k_2 + 4(2) - 7(1) = 4(k_1 + 2) + 5(k_2 - 1).$$

   Case 1. $k_2 \geq 1$. Then the above nonnegative linear combination suffices, since $(k_2 - 1) \geq 0$. **Case 2.** $k_2 = 0$. Then $n = 4k_1$, where $k_1 \geq 5$, since $n \geq 18$. Thus,

$$n + 1 = 4k_1 + 7(3) - 4(5) = 4(k_1 - 5) + 7(3)$$

   is a nonnegative linear combination of 4's and 7's. □

7. By checking the numbers 1 to 40, it appears that numbers $30 \leq n \leq 40$ are all nonnegative linear combinations of 4 and 11. Let's use induction to prove its true for all $n \geq 30$.

   Basis step $n = 30 = 4(2) + 11(2)$.

Inductive step Assume n is a nonnegative linear combination of 4's and 11's, for some $n \geq 30$. Show: $(n+1)$ is also a nonnegative linear combination of 4's and 11's. Since n is a nonnegative linear combination of 4's and 11's, we may write it as $n = 4k_1 + 11k_2$, where $k_1, k_2 \geq 0$. Then

$$n + 1 = 4k_1 + 11k_2 + 4(3) - 11(1) = 4(k_1 + 3) + 11(k_2 - 1).$$

Case 1. $k_2 \geq 1$. Then the above linear combination suffices, since $(k_2 - 1) \geq 0$. **Case 2.** $k_2 = 0$. Then $n = 4k_1$, where $k_1 \geq 8$, since $n \geq 30$. Thus,

$$n + 1 = 4k_1 + 11(3) - 4(8) = 4(k_1 - 8) + 11(3)$$

is a nonnegative linear combination of 4's and 11's. □

8. **Basis step** $n = 1$. If there is one puzzle piece, then $0 = 1 - 1$ moves are reqired.

   Inductive step Assume any puzzle having k pieces requires $k - 1$ moves, for all $k \leq n$, and for some $n \geq 1$.

   Show: A puzzle with $n + 1$ pieces requires n moves. Consider the final move required to assemble the puzzle. Thinking of a piece as a block of size 1, it will involve joining two blocks, one of size k, and the other of size $n + 1 - k$, for some $1 \leq k \leq n$. Then we also have $n + 1 - k \leq n$. Then by the inductive assumption, the size-k block required $k - 1$ moves to assemble, while the size-$(n+1-k)$ block required $n - k$ moves. Combining these moves, along with the final move, gives

$$(k - 1) + (n - k) + 1 = n$$

   moves. □

9. **Basis step 1** $n = 1 = 1 \bmod 4$. If there is one match, then the final match is removed by Player 1, and Player 2 wins.

   Basis step 2 If there are $n = 2, 3$ or 4 matches, then Player 1 removes $n - 1$ matches, leaving one for Player 2 who loses.

   Inductive step Assume any Nim game with $k \leq n$ matches is won by Player 1 when $k \bmod 4 \neq 1$, and by Player 2 when $k \bmod 4 = 1$. Show: a Nim game with $n + 1$ matches is won by Player 1 when $n + 1 \bmod 4 \neq 1$, and by Player 2 when $n + 1 \bmod 4 = 1$. **Case 1:** $n + 1 \bmod 4 = 0$. Then Player 1 should remove 3 matches, to yield $n - 2 < n$ matches, where $n - 2 \bmod 4 = 1$. Thinking of this as a new game with $n - 2$ matches, in which Player 1 now has the second move. It follows by the inductive assumption that the second player, namely Player 1, will win this game. **Case 2:** $n + 1 \bmod 4 = 2$. Same as Case 1, but now Player 1 removes 1 match. **Case 3:** $n + 1 \bmod 4 = 3$. Same as Case 1, but now Player 1 removes 2 matches. **Case 4:** $n + 1 \bmod 4 = 1$. In this case notice that, unlike the other cases, Player 1 cannot make a first move that leaves a number matches that, when divided by 4, has a remainder equal to 1. For this reason, the first player of the new reduced game will win. But this player is Player 2. □

10. Prove that $\sqrt{2} \neq n/b$ for any $n \geq 1$, and any fixed integer $b \geq 1$.

    Basis step $n = 1$. $\sqrt{2} \neq 1/b$, since $\sqrt{2} > 1$ and $1/b \leq 1$.

Inductive step Assume $\sqrt{2} \neq k/b$, for all $k \leq n$, and for any fixed constant $b \geq 1$. Show: $\sqrt{2} \neq (n+1)/b$, for any fixed $b \geq 1$. **Case 1.** $(n+1)/b$ is a reduced fraction. Then squaring both sides and moving b^2 to the left side yields

$$2b^2 = (n+1)^2,$$

which implies $n+1$ is even. Setting $n+1 = 2k$, this in turn implies

$$2b^2 = 4k^2 \Rightarrow 2k^2 = b^2,$$

which implies b is even. Thus $\gcd(n+1, b) \geq 2$, a contradiction. **Case 2.** $(n+1)/b$ is not a reduced fraction. Then we have $(n+1)/b = k/b'$, for some $b' \geq 1$ and some $k \leq n$. Hence, by the inductive assumption,

$$\sqrt{2} \neq k/b' = (n+1)/b.$$

□