Propositional Logic

By a constraint, we mean something that places limitations on what can represent a solution to a given problem, or on what can represent a valid state of some system. Moreover, a constraint is usually expressed in the form of a *rule*, a word that is synonymous with constraint. For example, the statement "Ronald only studies calculus on weekdays" represents a constraint that limits when Ronald can study. So if Ronald belongs to a calculus study group that only meets during times that are convenient to every member (another constraint!), then we can conclude that this group never meets on the weekends. Notice how rules stated like the one above lend themselves to being evaluated as being true or false. For this reason such a rule is referred to as a **logical proposition**, and is capable of being modeled using a form of logic known as **propositional logic**. For example, in propositional logic the statement made about Ronald may be expressed using a single **Boolean variable** named s, where $\texttt{dom}(s) = \mathcal{B} = \{\texttt{true},\texttt{false}\}$ is the set consisting of the Boolean values `true` and `false`. In other words, s may be assigned the value of `true` or `false`, since we believe that either Ronald does in fact only study calculus on the weekdays, or that there has been or will be an occurrence of Ronald studying calculus on the weekend. There is no "middle ground". Of course, there could be some gray area in terms of what it means to "study calculus". Does this include solving an interesting calculus problem on a Saturday morning, just for fun? Or how about mentally reviewing the definition of the Chain Rule of calculus just before bed on Sunday? Nevertheless, we assume that our definition of "studying calculus" allows us to answer these questions.

A single Boolean variable v is referred to as an **atomic proposition**, since it does not reduce further to other more basic propostions. On the other hand, a **compound proposition** is one that can be expressed using one or more atomic propositions, together with one or more *logical connectives*. For example, the statement "I'm either having the miso soup, or the tofu green salad" represents a compound proposition, since it uses logical exclusive-OR to connect two atomic propositions: "I'm having the miso soup", and "I'm having the tofu green salad". Here, exclusive-OR means that one of the statements must be true, but not both, since we assume that the speaker is implicitly indicating that she plans to select only one of the two dishes as part of her meal.

The exclusive-OR logical connective is just one of several that we now present in the following table.

Connective	**Symbol**	**Example**
NOT	$\neg$	“It is **not** raining outside.”
AND	$\wedge$	“It is raining **and** the sidewalk is wet.”
OR (Inclusive)	$\vee$	“A wrench **or** pliers is needed for the repair.”
OR (Exclusive)	$\oplus$	“He was born in either Phoenix **or** Tuscon.”
IF-THEN	$\rightarrow$	“**If** it rains **then** the sidewalk gets wet.”
EQUIVALENCE	$\leftrightarrow$	“You pass **if and only if** you study.”

The NOT connective acts on a single proposition, and asserts its falsehood. Each of the remaining connectives combines two propositions and asserts something in relation to the two. Indeed, AND asserts that both are true, inclusive-OR asserts that one of the two (or both) are true, exclusive-OR asserts that one of the two (but not both) are true, IF-THEN asserts that the first being true is a cause for the second to be true, and EQUIVALENCE asserts that either both are true, or both are false. Thus, with the exception of IF-THEN, the order in which the two propositions appear does not matter. For example, saying that “A wrench or pliers is needed for the repair” has the same truth as “Pliers or a wrench is needed for the repair” (although in practice one may want to first state the tool of preference). On the other hand, the statement “If the sidewalk is wet, then it is raining” has a different meaning than “If it is raining then the sidewalk is wet”, since the former is stating that the wetness of the sidewalk is enough to guarantee rain, which is not always true (think of sprinklers).

Example 1. Which of the following uses of language represents a proposition?

1. "Do you also like soup whenever you a salad?"
2. "Hand me either the pliers or wrench."
3. "If you place that fly on my soup, I'll scream!"
4. "Ahhhhhhh!"
5. "Kings Win!!!"

Example 2. Represent each of the following compound propositions with a propositional formula. Define all variables.

1. "I am not late"
2. "Although I study ten hours each week, still I am failing the class."
3. "I decided that next season I will sign with either the Lakers or the Jazz".

Truth Tables

Like the set operations defined in the Sets lecture, we may think of each binary logical connective as a means of taking two Boolean values a and b, and associating with them a third Boolean value c that is obtained by applying the connective to a and b. For example, if $a = \texttt{false}$ and $b = \texttt{true}$, then $a \wedge b = \texttt{false}$, since it is false that both a and b are true. For this reason each of the logical connectives is referred to as a **Boolean operation**, with NOT being a unary operation, and all others being binary operations. A unary Boolean operation only has two possible input values, namely `false` and `true`, while a binary operation has $2 \times 2 = 4$ different possible input combinations, namely

$$(\texttt{false}, \texttt{false}), (\texttt{false}, \texttt{true}), (\texttt{true}, \texttt{false}), (\texttt{true}, \texttt{true}).$$

Since there are only a handful of possible input combinations, we may show in a table the output of a Boolean operation in response to each of the input combinations. Such a table is referred to as a **truth table**.

For example, the following is a truth table for NOT.

p	$\neg p$
0	1
1	0

Notice that **false** is represented by 0 and **true** by 1. The first column gives the two possible truth values for some logical proposition p, while the second column gives the corresponding truth value of $\neg p$. This next truth table is for AND.

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

Here the first two columns provide the different truth combinations for the input propositions p and q, while third column gives the corresponding truth value of $p \wedge q$. Notice that $p \wedge q$ can only evaluate to 1 when both p and q evaluate to `true`. Table 1 shows the truth tables of the remaining Boolean operations combined into one table.

The $p \to q$ truth table seems worthy of comment. Recall from above that $p \to q$ is asserting that the truth of p is a cause for the truth of q. Here we refer to p as the **cause** and q as the **effect**. For example, if p represents "It's raining" and q represents "The sidewalk is wet", then $p \to q$ asserts that rain is a cause for a sidewalk getting wet. But it does *not* assert that rain is *the* cause for a sidewalk to get wet. Indeed, there could be no rain, but the sprinklers are causing the sidewalk to

p	q	$p \vee q$	$p \oplus q$	$p \to q$	$p \leftrightarrow q$
0	0	0	0	1	1
0	1	1	1	1	0
1	0	1	1	0	0
1	1	1	0	1	1

Table 1: Truth tables for some Boolean operations

a	b	c	$a \wedge b$	$(a \wedge b) \to c$
0	0	0	0	1
0	0	1	0	1
0	1	0	0	1
0	1	1	0	1
1	0	0	0	1
1	0	1	0	1
1	1	0	1	0
1	1	1	1	1

Table 2: Truth table for $(a \wedge b) \to c$

get wet. Thus $0 \to 1$ evaluates to `true`, since a wet sidewalk on a sunny day only establishes that there is more than one cause for a wet sidewalk, and in no way negates the fact that rain is one of the causes. Similarly, $0 \to 0$ evaluates to `true` since a dry sidewalk on a sunny day does nothing to disprove the causal link between between rain and wet sidewalks. In a system of logic that allows for more than two possible truth values, perhaps $0 \to 1 = 1$ would not be the preferred evaluation. But with only two choices, `true` or `false`, `false` seems too harsh a choice, and so we are left with `true`. An IF-THEN statement that is true because the cause is false is said to be **vacuously** true.

It's worth emphasizing that the p and q shown in the above truth tables can represent either atomic or compound propositions. For example, consider the compound propositional formula $(a \wedge b) \to c$. It has the form $p \to c$, where p is the **subformula** $a \wedge b$, i.e. a formula that occurs within the main formula. Now we could make a truth table for $(a \wedge b) \to c$ using the variables p and c, but it would look exactly like the one in Table 1, and would obfuscate the truth of the formula in relation to the original three variables a, b, and c. Moreover, there are a total of $2 \times 2 \times 2 = 8$ different truth-value combinations that can be assigned to these input variables, and so we need a table with eight rows. Table 2 shows the desired truth table. Notice that the table includes an additional column that gives the truth values for $(a \wedge b)$. This column serves as the "cause" column, while the c-column serves as the "effect" column which are both needed to compute the $((a \wedge b) \to c)$-column. For example, row 7 of the $((a \wedge b) \to c)$-column evaluates to 0 because the cause value in that row is 1 while the effect value is 0, giving $1 \to 0 = 0$.

Example 3. Make a truth table for the logical formula $(p \to \neg q) \oplus (p \wedge r)$.

a	b	$b \to a$	$a \to (b \to a)$
0	0	1	1
0	1	0	1
1	0	1	1
1	1	1	1

Table 3: Truth table for $a \to (b \to a)$

Logical Equivalence

Definition 1. Given propositional formula p, then p is a

1. **tautology** iff p always evaluates to `true`, regardless of how its variables are assigned.
2. **fallacy** iff p always evaluates to `false`, regardless of how its variables are assigned.
3. **contingency** iff p's variables may be assigned in a way that makes p evaluate to `true`, and may be assigned in another way that makes p evaluate to `false`.

Example 4. Let a and b be Boolean variables. Then

1. a is a contingency since $a = 0$ makes a false, and $a = 1$ makes a true.
2. $a \to b$ is a contingency since $(a = 1, b = 0)$ makes $a \to b$ false, while $(a = 1, b = 1)$ makes $a \to b$ true.
3. $a \vee \neg a$ is a tautology since, no matter how a is assigned, either a or $\neg a$ will evalute to `true`.
4. $a \wedge \neg a$ is a fallacy since, no matter how a is assigned, either a or $\neg a$ will evalute to `false`, and so a and $\neg a$ cannot both be true.
5. $a \to (b \to a)$ is a tautology. One way to see this is by constructing a truth table and verifying that the $a \to (b \to a)$-column consists of all 1's (see Table 3). Alternatively, notice that the only way this formula could be false is if a is true and $b \to a$ is false. But $b \to a$ is false only when a is false, which conflicts with the requirement that a be true. Therefore, the formula must be tautology.
6. $(a \to b) \to (\neg a \to \neg b)$ is a contingency, since it's true for $(a = 0, b = 0)$, but false for $(a = 0, b = 1)$.

Two propositional formulas p and q are said to be **logically equivalent** iff the formula $p \leftrightarrow q$ is a tautology. In other words, q is true whenever p is true, and p is true whenever q is true. The most common way to show that two formulas are logically equivalent is to construct their respective truth tables and verify that the p-column of p's table is identical with the q-column of q's table.

Example 5. Proposition $a \to b$ is logically equivalent to $\neg b \to \neg a$. One way of seeing this is to construct a truth table for both formulas and verify that the $(a \to b)$-column is identical to the $(\neg b \to \neg a)$-column. Alternatively, notice that $a \to b$ can only be made false by the assignment $(a = 1, b = 0)$. Similarly, $\neg b \to \neg a$ can only be made false when $\neg a$ is false and $\neg b$ is true, i.e. can only be made false by the assignment $(a = 1, b = 0)$. Therefore, they are equivalent.

a	b	$a \to b$	$\neg b$	$\neg a$	$\neg b \to \neg a$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	1	1	0	0	1

Example 6. Show that $\neg(p \wedge q)$ is logically equivalent to $\neg p \vee \neg q$.

The table below shows the most important logical equivalences of propositional logic. These equivalences are also often referred to as *logical axioms* and *logical identities*. For the sake of readability, 1 represents `true`, 0 represents `false`, and $=$ represents $\leftrightarrow$. Notice that each equivalence has a corresponding **dual** equivalence that is obtained by replacing $\vee$ with $\wedge$ (or $\wedge$ with $\vee$) and 1 with 0 (or 0 with 1). For example, the dual of $p \vee 1 = 1$ is $p \wedge 0 = 0$. Notice also that these equivalences match the set identities from the Sets lecture, with $\vee$ corresponding to $\cup$, $\wedge$ to $\cap$, 1 to $\mathcal{U}$, and 0 to $\emptyset$. In the parlance of mathematics we say that, together with their respective operations, the set $\mathcal{L}$ of all logical propositions and the set $\mathcal{S}$ of sets are both examples of a mathematical structure called a **Boolean algebra**, and the table below lists the axioms (expressed in logical form) of such an algebra.

Equivalence	**Name**
$p \wedge 1 = p$ $p \vee 0 = p$	Identity
$p \vee 1 = 1$ $p \wedge 0 = 0$	Domination
$p \vee p = p$ $p \wedge p = p$	Idempotency
$\neg(\neg p) = p$	Double negation
$p \vee q = q \vee p$ $p \wedge q = q \wedge p$	Commutativity
$(p \vee q) \vee r = p \vee (q \vee r)$ $(p \wedge q) \wedge r = p \wedge (q \wedge r)$	Associativity
$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$	Distributivity
$\neg(p \vee q) = \neg p \wedge \neg q$ $\neg(p \wedge q) = \neg p \vee \neg q$	De Morgan

More on Conditional Propositions

A proposition of the form $p \rightarrow q$ is called a **conditional** proposition, and appears quite often within natural language. When p and q are often respectively referred to as the **cause** and **effect**, or the **hypothesis** and **conclusion**.

Definition 2. Given the proposition $p \rightarrow q$,

1. $q \rightarrow p$ is called its **converse**
2. $\neg q \rightarrow \neg p$ is called its **contrapositive**
3. $\neg p \rightarrow \neg q$ is called its **inverse**

Example 7. Given the statement "If it's raining, then the sidewalk is wet", we have $p =$ "it's raining" and $q =$ "the sidewalk is wet". Then the converse reads "If the sidewalk is wet then it's raining", the contrapositive reads "if the sidewalk is dry, then it's not raining", while the inverse reads "if it's not raining, then the sidewalk is dry".

Proposition 1. The proposition $p \rightarrow q$ is logically equivalent to its contrapositive, while its converse is logically equivalent to its inverse.

Proof of Proposition 1. The logical equivalence of $p \rightarrow q$ and $\neg q \rightarrow \neg p$ was demonstrated in Example 5. Finally, the converse $q \rightarrow p$ is logically equivalent to the inverse $\neg p \rightarrow \neg q$, since the inverse is the contrapositive of the converse. $\square$

The following are some of the most common ways that a conditional statement $p \rightarrow q$ may be expressed in English.

p **implies** q**:** "Tingling of the tongue implies a vitamin B-12 deficiency."

if p**,** q**:** "If you choose not to decide, you still have made a choice."

p **only if** q**:** "You will pass this class only if you study."

p **is sufficient for** q**:** "Scoring an A on all exams is sufficient for receiving an A in the class."

q **if** p**:** "A number n is prime if it is not divisible by a number between 2 and $\lfloor \sqrt{n} \rfloor$."

q **whenever** p**:** "We will leave whenever Iqbal arrives."

q **is necessary for** p**:** "Studying outside of class at least five hours a week is necessary for passing this class."

Of the above examples, perhaps the two most misunderstood are " p only if q" and "q is necessary for p". Both of these have essentially the same meaning. For example, "p only if q" means that the only way p can be true is if q is true. In other words, if q is false, then so is p, i.e. $\neg q \rightarrow \neg p$ which is logically equivalent to $p \rightarrow q$. Thus, if $p =$ "you pass the class" and $q =$ "you study", then the statement "You will pass this class only if you study" is indicating that studying is not necessarily sufficient for passing, but if you pass the class, then certainly you studied. This may seem odd, since studying is the "effect" even though it precedes passing the class on the timeline. For this reason, it seems more natural to write the statement as "studying is necessary for passing the class".

Example 8. Provide a compound propositional formula that models the statement "For you to be hired, it is necessary that you are 18 years or older, and you have a high-school diploma".

Solution.

Example 9. Provide a compound propositional formula that models the statement "Of the four friends Al, Bo, Cindy, and Dan, exactly two of them program in Python".

Solution.

Exercises

1. Which of the following are propositions? Explain.

   a. I'll find employement when the economy improves.
   b. Please send me an application.
   c. They have decided to hire either Raymond or Terry.
   d. Would you be interested in hiring me?
   e. I'm feeling very nervous about the interview.
   f. Yes, I have had a course in discrete mathematics.
   g. The statement $x = 2y$ is not a proposition.

2. Is the statement "This sentence is false." a proposition? Explain.

3. Which of the following propositions are compound? Explain. For each compound proposition, provide a logical expression that models the statement. Define each propositional variable that is used.

   a. Robert bought a new IPhone.
   b. It's not true that Robert bought a new IPhone.
   c. John made a down payment of 10K.
   d. Having a down payment of at least 5K is necessary for buying this home.
   e. Bill will either travel to San Francisco by train or by car.
   f. It is not true that studying hard is sufficient for passing the class.
   g. Laura studied hard for CECS 174.
   h. To be accepted you must have at least a 3.0 GPA or at least an 1800 SAT score.
   i. Having the highest security ranking within the company is equivalent with working here for at least 10 years, and having no law-enforcement infractions.

4. How many distinct truth tables are there for Boolean functions that depend on two variables? For Boolean functions that depend on $n \geq 1$ variables?

5. Provide a propositional formula which has the following truth table.

p	q	
0	0	1
0	1	1
1	0	0
1	1	0

6. Provide a propositional formula which has the following truth table.

p	q	
0	0	1
0	1	1
1	0	1
1	1	0

7. Provide a truth table for $p \oplus (q \oplus r)$.

8. Recall that a subformula of a propositional formula F is any propositional formula that occurs in F. List the subformulas of the propositional formula $\neg(p \rightarrow (\neg q \rightarrow r))$.

9. Provide a truth table for the propositional formula $p \rightarrow (\neg q \rightarrow r)$. Include a column for each subformula.

10. Classify each of the following as a tautology, fallacy, or contingency.

    a. $P \rightarrow P$

    b. $P \rightarrow \neg P$

    c. $P \wedge (P \rightarrow \neg P)$

11. Is $(\neg p \wedge (p \rightarrow q)) \rightarrow \neg q$ a tautology? Explain and show work.

12. Is $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$ a tautology? Explain and show work.

13. Use a truth table to show that $p \rightarrow (q \vee r)$ is logically equivalent to $(p \rightarrow q) \vee (p \rightarrow r)$.

14. Use a truth table to show that $p \leftrightarrow (q \leftrightarrow r)$ is logically equivalent to $(p \leftrightarrow q) \leftrightarrow r$.

15. Write statements in English that represent the converse, contrapositive, and inverse of the statement "hard work is necessary for passing this class."

16. Write the converse, contrapositive, and inverse of the statement "he will win the election only if he wins Florida".

17. Show that the converse of a conditional statement is logically equivalent with its inverse.

18. Either Paul or Quan (or both) will attend CSULB. If Paul attends, then so will Sam. But Sam will not attend if his brother Robert attends. Finally, it was learned that Sam did not attend CSULB. Define propositional variables and model each of the given facts with a propositional formula.

19. Four friends have been identified as suspects for a burglary that police believe to be committed by only one person. They made the following statements to the police. Alice said, "Carlos did it". John said, "I did not do it". Carlos said, "Diana did it". Diana said, "Carlos lied when he said that I did it". Define propositional variables so that you can model these statements, including variables that indicate whether or not a given statement is true. Use them to also model the statement "exactly one of the four suspects is telling the truth".

20. Ada, Bea, and Clara have different characteristics. Exactly two are artistic; exactly two are beautiful; exactly two are intelligent; and exactly two are rich. Each has at most three characteristics. If Ada is intelligent, then she is also rich. Of Bea and Clara, it is true that, if she is beautiful, then she is artistic. Of Ada and Clara it is true that, if she is rich, then she is artistic.

    a. Define a set of propostional variables which encode whether or not a woman has a given characteristic. For example, aa might represent the proposition that Ada is artistic. How many variables are needed?

b. Use the variables from a. to provide a propositional formula that represents the statement "exactly two women are artistic".

c. Use the variables from a. to provide a propositional formula that represents the statement "Ada has at most three characteristics".

d. Use the variables from a. to provide a propositional formula that represents the statement "if Clara is rich, then she is artistic".

Exercise Answers and Hints

1. 1,3,5,6, and 7 are propositions.

2. No, because the statement neither evaluates to `true` nor `false` (why?)

3. b. $\neg p$; d. bh $\rightarrow$ dp; e. tsft $\oplus$ tsfc; f. $\neg(s \rightarrow p)$; h. hpga $\vee$ hsat $\rightarrow a$; i. hsr $\leftrightarrow$ wty $\wedge \neg$li

4. 16, 2^{2^n}

5. $\neg p$

6. $\neg(p \wedge q)$

7.

p	q	r	$q \oplus r$	$p \oplus (q \oplus r)$
0	0	0	0	0
0	0	1	1	1
0	1	0	1	1
0	1	1	0	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	0	1

8. p, q, r, $\neg q$, $\neg q \rightarrow r$, $p \rightarrow (\neg q \rightarrow r)$, $\neg(p \rightarrow (\neg q \rightarrow r))$

9. Provide a truth table for the propositional formula $p \rightarrow (\neg q \rightarrow r)$. Include a column for each subformula.

p	q	r	$\neg q$	$\neg q \rightarrow r$	$p \rightarrow (\neg q \rightarrow r)$
0	0	0	1	0	1
0	0	1	1	1	1
0	1	0	0	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	1	1
1	1	0	0	1	1
1	1	1	0	1	1

10. a: tautology; b: contingency; c: fallacy

11. Not a tautology: set p to false and q to true.

12. Yes. Construct a truth table.

13. Logically equivalent since columns 5 and 8 are identical.

p	q	r	$q \vee r$	$p \to (q \vee r)$	$p \to q$	$p \to r$	$(p \to q) \vee (p \to r)$
0	0	0	0	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	1	1	1	1
0	1	1	1	1	1	1	1
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	1	1
1	1	1	1	1	1	1	1

14. Logically equivalent since columns 5 and 7 are identical.

p	q	r	$q \leftrightarrow r$	$p \leftrightarrow (q \leftrightarrow r)$	$p \leftrightarrow q$	$(p \leftrightarrow q) \leftrightarrow r$
0	0	0	1	0	1	0
0	0	1	0	1	1	1
0	1	0	0	1	0	1
0	1	1	1	0	0	0
1	0	0	1	1	0	1
1	0	1	0	0	0	0
1	1	0	0	0	1	0
1	1	1	1	1	1	1

15. Converse: hardwork is sufficient for passing this class. Contrapositive: if you don't work hard, then you won't pass this class. Inverse: if you don't pass this class, then by conclusion you did not work hard.

16. Converse: if he wins Florida, then he will win the election. Contrapositive: if he doesn't win Florida, then he doesn't win the election. Inverse: if he doesn't win the election, then he doesn't win Florida.

17. Hint: make truth tables for each and show they are identical.

18. Variables: p,q,r, s, where, e.g., p is true iff Paul attends CSULB. Either Paul or Quan (or both) will attend CSULB: $p \vee q$ If Paul attends, then so will Sam: $p \to s$. But Sam will not attend if his brother Robert attends: $r \to \neg s$. Finally, it was learned that Sam did not attend CSULB: $\neg s$.

19. Hint: let ta be a variable that is true iff Alice is telling the truth. Let c be a variable that is true provided Carlos is the burglar. Then Alice's statement can be modeled as $\text{ta} \leftrightarrow c$.

20. Hint: if you wanted to show that exactly two women are artistic, then one possibility would be $\text{aa} \wedge \text{ba} \wedge \neg\text{ca}$, which states that both Ada and Bea are artistic, and Clara is not artistic. This is one possibility. But there are two other possibilities. Combine the three possibilities by taking their logical OR.