Logical Reasoning

Rules of Logical Inference

Deductive logical reasoning is the process of deriving propositional formulas from existing one that are assumed `true`. Moreover, these derived formulas must also be `true`, assuming the truth of the ones from which they are derived. Each derived formula is obtained with the help of an **inference rule**. Each inference rule has two parts: the assumption set, and the conclusion set. The **assumption set** is a set of propositional formulas that are all assumed `true`, where as the **conclusion set** is a set of propositional formulas that must also evaluate to `true`, assuming each formula in the assumption set are true. The following is a list of the most common and useful inference rules for propositional logic.

Simplification

Assume $p \wedge q$

Conclude p, q

Conjunction

Assume p, q

Conclude $p \wedge q$

OR Inference

Assume $p \vee q$, $\neg p$

Conclude q

OR Inference

Assume $p \vee q$, $\neg q$

Conclude p

General OR Inference

Assume $p_1 \vee p_2 \vee \cdots \vee p_k$, $\neg p_1, \ldots, \neg p_{i-1}, \neg p_{i+1}, \ldots, \neg p_k$

Conclude p_i

Addition

Assume p

Conclude $p \vee q$

Addition

Assume p

Conclude $q \vee p$

Conditional Inference

Assume $p \rightarrow q$, p

Conclude q

Contrapositive Inference

Assume $p \rightarrow q$, $\neg q$

Conclude $\neg p$

XOR Inference

Assume $p \oplus q$, $\neg p$

Conclude q

XOR Inference

Assume $p \oplus q$, $\neg q$

Conclude p

XOR Inference

Assume $p \oplus q$, p

Conclude $\neg q$

XOR Inference

Assume $p \oplus q$, q

Conclude $\neg p$

Equivalence

Assume $p \leftrightarrow q$, p

Conclude q

Equivalence

Assume $p \leftrightarrow q$, q

Conclude p

Equivalence

Assume $p \leftrightarrow q$, $\neg p$

Conclude $\neg q$

Equivalence

Assume $p \leftrightarrow q$, $\neg q$

Conclude $\neg p$

Example 1. What additional information, if any, can be inferred from each of the following statements?

- John will attend only if Mary attends. Mary is not attending.

- Mary or John will attend the party. Mary is attending the party.

- You will have soup or salad, but not both. OK, you're not having the soup.

- If it rains then the sidewalk gets wet. The sidewalk got wet.

Example 1 Solution.

In addition to the above inference rules, one may also use the following identities that were stated in the Propositional Logic lecture. For each identity, if one side of the identity has been assumed or established as true, then the other side must also evaluate to `true`.

Equivalence	Name
$p \wedge 1 = p$ $p \vee 0 = p$	Identity
$p \vee 1 = 1$ $p \wedge 0 = 0$	Domination
$p \vee p = p$ $p \wedge p = p$	Idempotency
$\neg(\neg p) = p$	Double negation
$p \vee q = q \vee p$ $p \wedge q = q \wedge p$	Commutativity
$(p \vee q) \vee r = p \vee (q \vee r)$ $(p \wedge q) \wedge r = p \wedge (q \wedge r)$	Associativity
$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$	Distributivity
$\neg(p \vee q) = \neg p \wedge \neg q$ $\neg(p \wedge q) = \neg p \vee \neg q$	De Morgan

Example 2. Either rain or the sprinklers being turned on is sufficient for the side walk to get wet. But the sidewalk is dry. Show how De Morgan's rule is used to conclude that the sprinklers are turned off.

Example 2 Solution.

Suppose A is a set of propositional formulas, q is a single propositional formula, and V is the set of Boolean variables one which these formulas depend. Then we say that q is a **logical consequence** of A iff, for any truth assignment to the variables in V that makes every formula in A evaluate to `true`, the assignment also makes q evaluate to `true`. Moreover, an inference rule is said to be **valid** iff the conclusion C is always a logical consequence of the set of assumptions $\mathcal{A} = \{A_1, \ldots, A_k\}$. Finally, we say that a set of inference rules is **sound** iff each inference rule is valid.

There is a way to tell if an inference rule is valid. If $A_1, \ldots, A_k$ are the assumptions, and C is the conclusion, then the formula

$$A_1 \wedge A_2 \wedge \cdots \wedge A_k \to C$$

must be a tautology. In other words, make a truth table for $A_1, \ldots, A_k$, and C, and verify that the C column has a 1 whenever each of the A_i columns, $i = 1, \ldots, k$ has a 1.

Example 3. Verify that
$$(\neg Q \wedge (P \rightarrow Q)) \rightarrow \neg P$$
is a tautology. Hence, it is not valid to conclude $\neg P$ if one knows that $\neg Q$ is true, as well as $P \rightarrow Q$.

Example 3 Solution.

Logical Derivations

Given a set of propositonal formulas $A = \{p_1, \ldots, p_k\}$ that are assumed true, a **logical derivation of** q **assuming** A is a sequence of propositional formulas $f_1, f_2, \ldots, f_n$ where

1. $f_i = p_i$, $i = 1, \ldots, k$.

2. $q = f_n$ and

3. for all $i > k$, either f_i follows from a valid inference rule or logical identity involving one or more formulas in the set $\{f_1, \ldots, f_{i-1}\}$.

A logical derivation provides a **direct proof** that formula q is true whenever the formulas in A are assumed true.

Example 4. Let A denote the set of propositonal formulas listed below. Provide a logical derivation of $S \vee Q$ assuming formula in A is true.

1. $(P \vee Q) \to R$

2. $\neg Q \to S$

3. $\neg R$

Example 4 Solution.

Contradictions

A logical derivation is said to be **contradictory** iff there are two formulas, f_i and f_j, in the derivation for which one is the negation of the other. In this case we say that a contradiction has been derived.

Let $A = \{p_1, \ldots, p_k\}$ be a set of propositional formulas and V the set of variables on which the formulas in A depend. Then we say that A is a **consistent** set of formulas if iff truth values may be assigned to the variables V in such a way that all formulas in A evaluate to `true`. If no such truth values exists, then A is called an **inconsistent** set of formulas. In other words, A is inconsistent iff

$$p_1 \wedge p_2 \wedge \cdots \wedge p_k$$

is a fallacy.

Theorem 1. If a logical derivation is contradictory, then the assumption set A is **inconsistent**.

We prove Theorem 1 in a later lecture once we have studied the necessary proof techniques.

Example 5. Verify that $p \wedge \neg r$, $\neg q \vee r$, and $p \rightarrow q$ is an inconsistent set of formulas by making a truth table and verifying that their conjunction is a fallacy.

Example 6. Repeat Example 5, but now use Theorem 1 to demonstrate inconsistency by providing a contradictory derivation that has assumption set equal to

$$A = \{p \wedge \neg r, \neg q \vee r, p \rightarrow q\}.$$

$$A = \{p \wedge \neg r, \neg q \vee r, p \rightarrow q\}.$$

Proof by contradiction

A set of inference rules is said to be **complete** iff for every set of assumptions A and every formula q that is a logical consequence of A, q can be logicaly derived from A via a direct step-by-step proof. Otherwise, the set of rules is said to be **incomplete**. It turns out that the inference rules that we've introduced in this lecture are incomplete. In other words, there are times when a formula q is a logical consequence of a set of formulas A, but there is no way to derive q using A as the assumption set along with all the provided rules of inference. When this happens, we need to use other means to establish that q is a logical consequence of A. Of course, we may always use a truth table, but sometimes there are too many variables for this to be practical.

In what follows, let A be a consistent set of formulas. Another strategy for showing that q is a logical consequence of A is to make $A \cup \{\neg q\}$ the assumption set and derive a contradiction. This technique is called **proof by contradiction**. By Theorem 1, this would imply that $A \cup \{\neg q\}$ is an inconsistent set. But we have assumed that A is consistent, meaning that there is at least one variable assignment α that makes A true. Moreover, since $A \cup \{\neg q\}$ is inconsistent, it means that α makes $\neg q$ `false`, which means that it makes q `true`. Therefore, q evaluates to `true` whenever an assignment α makes each formula in A evaluate to `true`, and so we have proved that q is a logical consequence of p.

Note: if q is a logical consequence of A, then it does *not* necessarily mean that there exists a logical derivation of q whose assumption set is A (since the set of inference rules may be incomplete). However, the converse *is* true: if assumption set A derives q then q is a logical consequence of A.

Example 7. Let A denote the assumption set from Example 4. Show that $\neg(S \vee Q)$ is a logical consequence of A by assuming $A \cup \{\neg(S \vee Q)\}$ and deriving a contradiction.

1. $(P \vee Q) \to R$

2. $\neg Q \to S$

3. $\neg R$

4. $\neg(S \vee Q)$

Proof by Cases

Suppose we have a consistent set of formulas A, and we wish to show that q is a logical consequence of A. Suppose further that we have failed to directly derive q from A. Aside from a proof by contradiction, another technique that may be used is called **proof by cases**. Using this technique, we may add one or more additiional assumptions to A. For simplicity, in this lecture we assume that exactly one additional formula f is added to A and that one of two outcomes occur after adding f to A.

Case 1. by adding f to A we are now able to derive q. In this case we must now start over by now adding $\neg f$ to A and either deriving q or deriving a contradiction. If we derive q again, then it means that q is a logical consequence of A, since every assignment α that satisfies A must either satisfy f or $\neg f$ in which case, based on either of the two derivations, α will satisfy q. On the other hand, should we derive a contradiction, then f is a logical consequence of A, which makes q a logical consequence of A (explain why this is the case).

Case 2. by adding f to A we are now able to derive a contradiction. In this case we have learned that $\neg f$ is a logical consequence of A, and so $\neg f$ may be added as an additional assumption without jeapordizing consistency. We then start a new derivation and see if we are able to derive q. If yes, we are done. If no, we may repeat the proof-by-cases technique again.

Example 8a. Consider the following consistent set $\mathcal{A}$ of propositional formulas. Show that by adding formula G to this set, we are able to derive a contradiction.

1. $D \oplus G$

2. $G \to (\neg E \wedge D)$

3. $A \vee F \vee \neg G$

4. $A \leftrightarrow G$

5. $\neg C \to (A \vee B)$

6. $C \to (A \wedge \neg D)$

7. $A \vee C \vee E$

8. $B \oplus F$

Example 8a. Consider the following consistent set $\mathcal{A}$ of propositional formulas. Show that by adding formula $\neg G$ to this set, we are able to derive the formula

$$\mathcal{F} = \neg A \wedge B \wedge \neg C \wedge D \wedge E.$$

Conclude that $\mathcal{F}$ is a logical consequence of $\mathcal{A}$.

1. $D \oplus G$

2. $G \rightarrow (\neg E \wedge D)$

3. $A \vee F \vee \neg G$

4. $A \leftrightarrow G$

5. $\neg C \rightarrow (A \vee B)$

6. $C \rightarrow (A \wedge \neg D)$

7. $A \vee C \vee E$

8. $B \oplus F$

Example 9a. Knights always tell the truth while Knaves always lie. Assume Alice is either a Knight or a Knave. Assume the same about Bob. Alice says "Bob and I are both knaves". What can you conclude from this statement?

Example 9b. Use the argument from Example 9a as a guide to formally proving that the Example-9a conclusions are a logical consequence of the Example-9a assumptions.

Exercises

1. Consider the inference rule where one assumes p and $p \vee q$ are true, and then concludes that q must be true. Show that this inference rule is not valid.

2. Consider the inference rule where one assumes p and $p \leftrightarrow q$ are true, and then concludes that q must be true. Show that this inference rule is valid.

3. Let $A = \{(a \vee b) \rightarrow (c \vee d), \neg c, b\}$. Using any logical inferences or identities presented in this lecture, provide a logical derivation of d assuming A.

4. Let $A = \{(a \wedge b) \rightarrow (c \wedge d), \neg c, b\}$. Using any logical inferences or identities presented in this lecture, provide a logical derivation of $\neg a$ assuming A.

5. Either Paul or Quan (or both) will attend CSULB. If Paul attends, then so will Sam. But Sam will not attend if his brother Robert attends. Finally, it was learned that Sam did not attend CSULB.

    a. Define propositional variables p, q, r, and s and model each of the given facts with a propositional formula.

    b. Use the formulas from part a to derive $\neg p \wedge q$.

    c. Explain why Robert attending (or not attending) is not a logical consequence of the provided facts.

6. Suppose a, b, and c are Boolean variables.

    a. Provide a propositional formula that represents the statement "exactly one of a, b, and c is true".

    b. Use your answer from part a, along with the assumptions $\neg b$ and $\neg c$, to logically derive a. Hint: in addition to a few other inferences, you'll need four uses of De Morgan, two uses of Commutativity (see list of logical identities) and one use of generalized OR.

7. Abe, Ben, Carl, and Darin may or may not attend a baseball game. The following facts will determine who attends. i) Carl will attend provided Ben attends, ii) either Abe is not attending or Carl is not attending (note: inclusive OR being used), iii) either Abe or Darin is attending (again, inclusive OR being used), iv) if Darin attends then Ben will attend but Carl will not attend.

    a. Using Boolean variables a, b, c, and d, write propositional formulas that express each of the above facts.

    b. Show that if we assume d along with the above four facts, then we may derive a contradiction. Conclude that $\neg d$ is a logical consequence of the above four facts.

    c. Show that if we assume $\neg d$ along with the above four facts, then we may derive a single formula that indicates who is attending and who is not attending the game.

8. Translate the following clues about which of three friends Al, Ben, and Cris. are attending a movie: i) Cris is attending if either Al is attending or Ben is not attending (inclusive OR), ii) if Cris is attending then all three are attending, iii) if Al is attending, then at least one of the three is not attending, and iv) at least one of the three is attending.

a. Using Boolean variables a, b, and c, write propositional formulas that express each of the above facts.

b. Show that if we assume c along with the above four facts, then we may derive a contradiction. Conclude that $\neg c$ is a logical consequence of the above four facts.

c. Show that if we assume $\neg c$ along with the above four facts, then we may derive a single formula that indicates who is and who is not attending the movie.

9. Let A be a set of propositional formulas and suppose propositional formula f is a logical consequence of A. Furthermore, suppose formula g is a logical consequence of $A \cup \{f\}$, then prove that g is a logical consequence of A.

10. Alice says "Bob and I are both knights". Bob says, "Alice is a knave".

a. Use the variables ta (Alice is telling the truth), tb (Bob is telling the truth), ka (Alice is a knight), and kb (Bob is a knight) to model these statements.

b. Use the formulas from part a and the additional assumption ka to derive a contradiction.

c. Use the formulas from part a and the additional assumption $\neg ka$ to determine whether or not Bob is a knight.

11. Alice says "Bob and I are both knaves". Bob says nothing.

a. Use the variables ta (Alice is telling the truth), ka (Alice is a knight), and kb (Bob is a knight) to model these statements.

b. Use the formulas from part a and the additional assumption ta to derive a contradiction.

c. Use the formulas from part a and the additional assumption $\neg ta$ to determine whether or not Bob is a knight.

12. You approach two men, each of whom is standing next to a door. One door leads to room with a pot of gold, while the other leads to a deadly encounter with a python snake. You know that one man is a knight, while the other is a knave, but you do not know which is the knight. What one question should you ask (to either man) in order to determine the room with the gold? Note that you can ask only one of the men.

Exercise Answers and Hints

1. Show that $(p \land (p \lor q)) \to q$ is not a tautology.

2. Show that $(p \land (p \leftrightarrow q)) \to q$ is a tautology.

3. The following is a logical derivation.

    1. $(a \lor b) \to (c \lor d)$
    2. b
    3. $\neg c$
    4. $a \lor b$ (2)
    5. $c \lor d$ (1,4)
    6. d (3,5)

4. The following is a logical derivation.

    1. $(a \land b) \to (c \land d)$
    2. b
    3. $\neg c$
    4. $\neg c \lor \neg d$ (3)
    5. $\neg(c \land d)$ (4, De Morgan)
    6. $\neg(a \land b)$ (1,5)
    7. $\neg a \lor \neg b$ (6, De Morgan)
    8. $\neg a$ (2,7)

5.   a. $p \lor q, p \to s, r \to \neg s, \neg s$

    b.   1. $p \lor q$
          2. $p \to s$
          3. $r \to \neg s$
          4. $\neg s$
          5. r
          6. $\neg p$ (2,4, contrapos.)
          7. q (1,6, OR)
          8. $\neg p \land q$ (6,7,Conjunction)

    c. The only formula that makes reference to r is $r \to \neg s$. Since $\neg s$ is true, the conditional statement is true regardless of the value of r. Therefore, neither r nor $\neg r$ is forced to be true based on $\neg p$, q, and $\neg s$ being true.

6.   a. $(a \land \neg b \land \neg c) \lor (\neg a \land b \land \neg c) \lor (\neg a \land \neg b \land c)$

    b. We have the following derivation of a.

          1. $(a \land \neg b \land \neg c) \lor (\neg a \land b \land \neg c) \lor (\neg a \land \neg b \land c)$

2. $\neg b$

3. $\neg c$

4. $\neg b \lor \neg\neg(a \lor c)$ (2, Add)

5. $\neg(b \land \neg(a \lor c))$ (4 De Morgan)

6. $\neg(b \land \neg a \land \neg c)$ (5 De Morgan)

7. $\neg(\neg a \land b \land \neg c)$ (6 Commutativity)

8. $\neg c \lor \neg\neg(a \lor b)$ (3, Add)

9. $\neg(c \land \neg(a \lor b))$ (8 De Morgan)

10. $\neg(c \land \neg a \land \neg b)$ (9 De Morgan)

11. $\neg(\neg a \land \neg b \land c)$ (10 Commutativity)

12. $(a \land \neg b \land \neg c)$ (1,7,11, Geneeralized OR)

13. a (12, Simp.)

7.   a. $b \to c$, $\neg a \lor \neg c$, $a \lor d$, $d \to (b \land \neg c)$

     b.   1. $b \to c$

           2. $\neg a \lor \neg c$

           3. $a \lor d$

           4. $d \to (b \land \neg c)$

           5. d

           6. $(b \land \neg c)$ (4,5, Conditional)

           7. b (6, Simp.)

           8. c (1,7, Conditional)

           9. $\neg c$ (6, Simp.)

         Contradiction: 8,9 ❒

     c.   1. $b \to c$

           2. $\neg a \lor \neg c$

           3. $a \lor d$

           4. $d \to (b \land \neg c)$

           5. $\neg d$

           6. a (3,5, OR)

           7. $\neg\neg a$ (6, Double Negation)

           8. $\neg c$ (2,7,OR)

           9. $\neg b$ (1,8,Contrapositive)

           10. $a \land \neg b$ (6,9, Conjunction)

           11. $\neg c \land \neg d$ (5,8, Conjunction)

           12. $a \land \neg b \land \neg c \land \neg d$ (10,11, Conjunction) ❒

     Have fun going by yourself Abe!

8.   a. $(a \lor \neg b) \to c$, $c \to (a \land b \land c)$, $a \to (\neg a \lor \neg b \lor \neg c)$, $a \lor b \lor c$

     b.   1. $(a \lor \neg b) \to c$

           2. $c \to (a \land b \land c)$

3. $a \rightarrow (\neg a \vee \neg b \vee \neg c)$

4. $a \vee b \vee c$

5. Assume c

6. $a \wedge b \wedge c$ (2,5, Conditional)

7. a (6, Simp.)

8. $\neg a \vee \neg b \vee \neg c$ (3,7, Conditional)

9. $\neg\neg a$ (7, Double Negation)

10. $\neg\neg c$ (5, Double Negation)

11. $\neg b$ (8,9,10, Gen. OR)

12. b (6, Simp.)

Contradiction: (11,12) ☐

c.   1. $(a \vee \neg b) \rightarrow c$

2. $c \rightarrow (a \wedge b \wedge c)$

3. $a \rightarrow (\neg a \vee \neg b \vee \neg c)$

4. $a \vee b \vee c$

5. Assume $\neg c$

6. $\neg(a \vee \neg b)$ (1,5, Contrapos.)

7. $\neg a \wedge \neg\neg b$ (6, De Morgan)

8. $\neg a$ (7, Simp.)

9. $\neg\neg b$ (7, Simp.)

10. b (9, Double Negation)

11. $\neg a \wedge b$ (9,10, Conjunction)

12. $\neg a \wedge b \wedge \neg c$ (5,11, Conjunction) ☐

Have fun going by yourself Ben!

9. Let α be an assignment that makes all the formula in A to evaluate to `true`, then since f is a logical consequence of A, f must also evaluate to `true`. Finally, since g is a logical consequence of $A \cup \{f\}$, g must also evaluate to `true`. Therefore, since α was arbitrary, g is a logical consequence of A.

10.   a. $ta \leftrightarrow ka$, $tb \leftrightarrow kb$, $ta \leftrightarrow (ka \wedge kb)$, $tb \leftrightarrow \neg ka$

b.   1. $ta \leftrightarrow ka$

2. $tb \leftrightarrow kb$

3. $ta \leftrightarrow (ka \wedge kb)$

4. $tb \leftrightarrow \neg ka$

5. Assume ka

6. ta (1,5, Equiv)

7. $(ka \wedge kb)$ (3,6, Equiv)

8. kb (7, Simp)

9. tb (2,8, Equiv)

10. $\neg ka$ (4,9, Equiv)

Contradiction: (5,10) ☐

c. 1. $ta \leftrightarrow ka$
2. $tb \leftrightarrow kb$
3. $ta \leftrightarrow (ka \wedge kb)$
4. $tb \leftrightarrow \neg ka$
5. Assume $\neg ka$
6. tb (4,5, Equiv)
13. kb (2,6,Equiv)
14. $\neg ta$ (1,5, Equiv)
Conclusion: Alice is a knave and Bob is a knight which is consistent with statements 1-4. ❐

11. a. $ta \leftrightarrow ka$, $ta \leftrightarrow (ka \wedge kb)$

b. 1. $ta \leftrightarrow ka$
2. $ta \leftrightarrow (\neg ka \wedge \neg kb)$
3. Assume ta
4. $(\neg ka \wedge \neg kb)$ (1,3,Equiv)
5. $\neg ka$ (4, Simp.)
6. $\neg ta$ (1,5,Equiv)
Contradiction: (3,6) ❐

c. '
1. $ta \leftrightarrow ka$
2. $ta \leftrightarrow (\neg ka \wedge \neg kb)$
3. Assume $\neg ta$
4. $\neg(\neg ka \wedge \neg kb)$ (2,3,Equiv)
5. $\neg\neg ka \vee \neg\neg kb$ (5, De Morgan)
6. $\neg ka$ (1,3,Equiv)
7. $\neg\neg\neg ka$ (6, Double Negation)
12. $\neg\neg kb$ (5,7,OR)
13. kb (12, Double Negation)
Conclusion: Alice is a knave and Bob is a knight which is consistent with statements 1-3. ❐

12. Hint: ask one of the men a question that pertains to the other man.