Sets and Set Operations

A **set** represents a collection of items, where each item is called a **member** of the set.

The most common way to represent a set is by using **list notation**, where the set members are listed one-by-one, and the list is delimited by braces. For example,

$$\{2, 3, 5, 7, 11\}$$

uses list notation to describe the set consisting of all prime numbers that do not exceed 11. Note that the order in which the members are listed does not matter. Indeed the sets $\{2, 3, 5, 7, 11\}$ and $\{3, 11, 5, 2, 7\}$ are identical. Also, each member occurs only *once* in the set, meaning that, e.g. $\{1, 2, 2, 3, 3, 3\} = \{1, 2, 3\}$.

The set having no members is called the **empty set**, and is denoted by $\emptyset$.

It is often convenient to provide a name or label for a given set. For example, $\mathcal{I}$ is used to denote the set whose members are the numbers $0, \pm 1, \pm 2, \ldots$. This book uses the convention that set names begin with a capital letter. Moreover, capital letters $A, B, C, \ldots$ are used when denoting an *abstract set*, meaning some arbitrary set whose members have not been specified. In later sections and chapters we shall see several other methods for naming and representing a set.

We may use the **membership symbol** $\in$ to indicate that an item is a member of some set. For example, $-2 \in \mathcal{I}$ asserts that -2 is a member of $\mathcal{I}$. On the other hand, $3.14 \notin \mathcal{I}$ asserts that 3.14 is *not* a member of $\mathcal{I}$ since it is not an integer.

Sometimes we need to use **informal list notation** by including the **ellipsis** $\ldots$ to indicate that a pattern is to be continued up to some value. The beginning pattern usually consists of one or more members that possibly make an "obvious" pattern, followed by an ellipsis, and ending with zero or more members of the set.

Example 1. Use informal list notation to describe i) the set of integers between 1 and 100 that are divisible by 3, and ii) the set of even integers.

Solution.

Set cardinality

The **cardinality** of set A, denoted $|A|$, is equal to the number of members of A. A set may be classified based on its type of cardinality.

Finite a set of the form $A = \{s_1, s_2, \ldots, s_n\}$, in which case $|A| = n$

Infinite a set that is not finite

Countably Infinite an infinite set whose members can be written in a never-ending list. In this case $|A| = \infty$

Uncountably Infinite a set that is not finite and whose members cannot be written in a never-ending list.

Example 2. The set of **natural numbers** $\mathcal{N} = \{0, 1, 2, \ldots\}$ is countably infinite since its members may be listed as $0, 1, 2, \ldots$. $\square$

Example 3. Show that the set of integers $\mathcal{I}$ is countably infinite.

Solution.

The Dovetailing procedure for listing a countably infinite set

A rational number is any number that may be written as a fraction p/q, where p and q are integers, and $q \neq 0$. Perhaps somewhat surprising is that the set $\mathcal{Q}$ of rational numbers is countably infinite. To show this we use a method called **dovetailing**, where members of $\mathcal{Q}$ are listed in rounds. In particular, i members of $\mathcal{Q}$ are listed in round i, $i = 1, 2, \ldots$. In what follows we show how the positive rationals can be listed with dovetailing. In round 1 we list $1/1$. In round 2 we list $1/2, 2/1$. In round 3 we list $1/3, 2/2, 3/1$. In round 4 we list $1/4, 2/3, 3/2, 4/1$. See the pattern? To generalize, in round i we list $1/i, 2/(i-1), 3/(i-2), \ldots, i/1$.

There are two basic questions that we would like to answer about the dovetailing procedure for rational numbers. For what value of j is p/q the j th fraction listed? For example, $1/1$ is the first (i.e. $j = 1$) fraction listed. Also, given positive j what is the j th fraction listed?

To answer the first question, notice that, after q rounds, a total of

$$1 + 2 + \cdots + q = q(q+1)/2$$

fractions are listed, and $1/q$ is listed in round q. Furthermore, p/q gets listed after an additional $p-1$ rounds, and so p/q is listed in round $p+q-1$, and it is the p th fraction listed. Thus p/q is the j th fraction listed, where

$$j = (p+q-2)(p+q-1)/2 + p.$$

Now suppose we have been given j and want to know of the j th fraction listed. The first step is to determine the penultimate round r, i.e. the round for which the j th fraction is listed *after* this round. We do this by solving the quadratic equation

$$x(x+1)/2 = j$$

and rounding down the equation's positive solution (if it is not an integer), or subtracting 1 from it if it is an integer. Letting r denote this rounded down integer, we see that the desired fraction has numerator

$$p = j - r(r+1)/2$$

and denominator

$$q = r - p + 1.$$

Example 4. Using the dovetailing technique of the previous example, for what value of $j \geq 1$ is $51/137$ the j th fraction listed?

Solution. The fraction $51/137$ is listed in round $51 + 137 - 1 = 187$. This is because $1/137$ appears in round 137, and 50 more rounds are needed for $51/137$ to be listed. Also, the total number of fractions listed in rounds 1 through 186 equals

$$1 + 2 + 3 + \cdots + 186 = (186)(187)/2 = (93)(187) = 17,391.$$

Finally, $51/137$ is the 51st fraction listed in round 187. Therefore, overall it is the $17,391 + 51 = 17,442$ nd fraction listed. $\square$

Example 5. Using the dovetailing technique of the previous example, what fraction will be the 100th fraction listed?

Solution. If the 100th fraction occurred at the very end of round n, then we would have $n(n+1)/2 = 100$, and solving this equation for n gives $n = 13.65$. This means that the penultimate round is $\lfloor 13.65 \rfloor = 13$. Moreover, $13(14)/2 = 91$ fractions have already been listed in rounds 1 through 13. Thus we need the 9th fraction listed in round 14. And this fraction is $9/6 = 9/(14 - 9 + 1)$. $\qquad \square$

For the remainder of the book, $\mathcal{N}$ denotes the natural numbers $\{0, 1, 2, \ldots\}$, $\mathcal{I}$ the integers, $\mathcal{Q}$ the rational numbers, and $\mathcal{R}$ the real numbers. Also, the addition of a plus, such as $\mathcal{R}^+$, denotes the positive members of a numerical set, while the addition of a minus, such as $\mathcal{I}^-$ denotes the negative members of a numerical set.

Subsets

Set A is said to be a **subset** of set B iff every member of A is also a member of B. This is symbolically denoted by $A \subseteq B$. Moreover, A is said to be a **proper subset** of B, denoted $A \subset B$, iff it is a subset of B, but there is some member of B who is not a member of A.

Example 6. Sets $\{3\}$ and $\{1,3\}$ are proper subsets of $\{1,2,3\}$, while $\{1,2,3\}$ is a subset of $\{1,2,3\}$, but not a proper subset. Also, is $A = \{\{1,2\},\{3,4\},5\}$ a subset of $B = \{1,2,3,4,5\}$? No since, e.g., $\{1,2\} \in A$ (yes, a set can be a member of another set!), but $\{1,2\} \notin B$. This is true since $\{1,2\}$ is a set, and B's members are the natural numbers 1 through 5.

Example 7. Given $A = \{1,3,5,\{2,4\},\{3,6,7\}\}$ and $B = \{2,\{3,7\},3,4,\{5\},\{6,8,9\}\}$,Which of the following statements are true?

    a. $\{2,4\} \subseteq B$.

    b. $\{3,7\} \subseteq B$.

    c. $9 \in B$.

    d. $\{5\} \in B$.

Example 8. List all subsets of the set $\{\emptyset, 1, \{2, 3\}\}$.

Solution.

The **power set** of a set S, denoted $\mathcal{P}(S)$, is defined as the set of all subsets of set S.

Example 9. For $S = \{1, 2, 3\}$ we have

$$\mathcal{P}(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Proposition 1. If S is a finite set with $|S| = n$, then $|\mathcal{P}(S)| = 2^n$.

Proof of Proposition 1. Without loss of generality, we may assume that $S = \{1, 2, \ldots, n\}$. Let A be a subset of S and consider the statements $i \in A$, for $i = 1, 2, \ldots, n$. Each of these statements will either evaluate to true (1) or false (0), depending on the choice of i and A. For example, if $A = \{1, 3, 5\}$ and $i = 2$, then the statement $2 \in A$ evaluates to 0, since 2 is not a member of A. Placing these statement evaluations side-by-side, we get the binary sequence

$$(1 \in A, 2 \in A, \ldots, n \in A).$$

For example, if $n = 5$ and $A = \{1, 3, 5\}$, then

$$(1 \in A, 2 \in A, 3 \in A, 4 \in A, 5 \in A) = (1, 0, 1, 0, 1).$$

Conversely, every binary sequence of length n corresponds with a subset of S. For example, if $n = 5$, then $(0, 0, 1, 0, 0)$ corresponds with subset $\{3\}$, while $(1, 1, 1, 0, 1)$ corresponds with $\{1, 2, 3, 5\}$. Therefore, the number of subsets of S is equal to the number of binary sequences having length n, and the reader may check that there are 2^n such sequences. As an example, the following **subset table** shows the correspondence between binary sequences of length three and subsets of $S = \{1, 2, 3\}$.

$1 \in A$	$2 \in A$	$3 \in A$	Subset A
0	0	0	$\emptyset$
0	0	1	$\{3\}$
0	1	0	$\{2\}$
0	1	1	$\{2, 3\}$
1	0	0	$\{1\}$
1	0	1	$\{1, 3\}$
1	1	0	$\{1, 2\}$
1	1	1	$\{1, 2, 3\}$

$\square$

Set operations

A binary arithmetic operation represents a way of taking two numbers, a and b, and associating with them a third number c that is obtained by performing the operation on a and b. For example, the binary addition operation associates with a and b the number $c = a + b$. In what follows we introduce some analogous binary set operations. Each of these operations takes two sets, A and B, and associates with them a third set in accordance with some rule of operation.

Union $x \in A \cup B$ iff either $x \in A$ or $x \in B$

Intersection $x \in A \cap B$ iff $x \in A$ and $x \in B$

Difference $x \in A - B$ iff $x \in A$ and $x \notin B$

Symmetric Difference $x \in A \oplus B$ iff $x \in A$ or $x \in B$, but not both; i.e. $A \oplus B = (A - B) \cup (B - A)$

Cartesian Product $(a, b) \in A \times B$ iff $a \in A$ and $b \in B$

Notice that for the Cartesian product, a member of $A \times B$ has the form (a, b), which is called a (binary) **tuple**, while a and b each represent a **component** of the tuple. Note that the term "tuple" is synonymous with several others in math and computing, including *array*, *vector*, *list*, and *sequence*. These tuples of $A \times B$ represent all possible ways of combining a member of A with a member of B. Also, the Cartesian product may be generalized to an arbitrary number of sets, such as $A_1 \times A_2 \times \cdots \times A_k$, in which case the members of this set are k-tuples of the form $(a_1, a_2, \ldots, a_k)$, such that $a_i \in A_i$, for all $i = 1, 2, \ldots, k$.

Example 10. Given sets $A = \{1, 3, 5, 6, 7, 9\}$ and $B = \{0, 2, 4, 5, 6, 7, 8, 10\}$, we have $A \cup B = \{0, 1, 2, \ldots, 10\}$, $A \cap B = \{5, 6, 7\}$, $A - B = \{1, 3, 9\}$, and $A \oplus B = \{0, 1, 2, 3, 4, 8, 9, 10\}$.

Example 11. Let $\mathcal{E}$ denote the set of all even integers. Compute $\mathcal{N} \cup \mathcal{E}$, $\mathcal{N} \cap \mathcal{E}$, $\mathcal{N} - \mathcal{E}$, $\mathcal{E} - \mathcal{N}$, and $\mathcal{N} \oplus \mathcal{E}$.

Solution.

Example 12. Given sets $A = \{a, b\}$, $B = \{1, 2, 3\}$, then $A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$.

In addition to the above binary operations, there is one important unary operation called the **complement** of set A, and denoted $\overline{A}$. In particular, $x \in \overline{A}$ iff $x \notin A$. The complement is only useful in cases when there is a pre-defined **universe** $\mathcal{U}$ consisting of all possible items that could appear in any set. In this case we have $\overline{A} = \mathcal{U} - A$. For example, if the universe $\mathcal{U}$ is given as the integers from 1 to 10, and $A = \{1, 3, 4, 8\}$, then $\overline{A} = \{1, 2, 5, 7, 9, 10\}$.

We say that two sets A and B are **disjoint** iff $A \cap B = \emptyset$. For example, the set of even integers and the set of odd integers are disjoint, since no integer is both even and odd.

The following is a table of identities that are always true for any three sets A, B, and C.

Identity	Description
$A \cup \emptyset = A$ $A \cap \mathcal{U} = A$	Identity
$A \cup \mathcal{U} = \mathcal{U}$ $A \cap \emptyset = \emptyset$	Domination
$\overline{\overline{A}} = A$	Double Complement
$A \cup A = A$ $A \cap A = A$	Idempotent
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative
$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$	Associative
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (B \cap C)$	Distributive
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan

A **membership table** is a table that can be used to verify a set identity of the form $S_1 = S_2$, where S_1 and S_2 are set expressions that depend on a finite number of arbitrary sets, meaning each set can be any possible set. The rows of the table consist of binary vectors that encode the membership (or lack of membership) of an arbitrary item x in each of the arbitrary sets that comprise the identity. For example, if the identity uses arbitrary sets A, B, and C, then the binary vectors will have a length equal to three, and a vector such as (0 1 0), would indicate $x \notin A$, $x \in B$, and $x \notin C$. Using this vector, we may compute $x \in S_1$ and $x \in S_2$, and verify that the values are equal. If they are ever unequal, then $S_1 = S_2$ is *not* an identity. However, if they are always equal for all $2^3 = 8$ different vector encodings, then $S_1 = S_2$ is in fact an identity.

Example 13. Verify the first distributive axiom using a membership table

Example 14. Verify that $A - (B \cap C) = (A - B) \cap (A - C)$ is *not* an identity.

Exercises

1. Which of the following pairs of sets are equal?

   a. $\{1, 3, 3, 3, 5, 5, 5, 5, 5\}$, $\{1, 3, 5\}$

   b. $\{\{1\}\}$, $\{1, \{1\}\}$

   c. $\emptyset$, $\{\emptyset\}$

2. Use list notation to express the set of positive integers that are perfect squares and do not exceed 100.

3. Use list notation to express the set of nonnegative integers that are less than 7.

4. Use list notation to express the set of rational numbers x for which $x^2 = 2$.

5. Use informal list notation to express the set of fractions of the form $1/n$, where n is an even positive integer between 1 and 1000 (inclusive).

6. Use informal list notation to express the set of perfect squares, i.e., nonnegative integers x for which there is an integer y such that $y^2 = x$.

7. Given $A = \{2, 4, 6\}$, $B = \{2, 6\}$, $C = \{4, 6\}$, and $D = \{4, 6, 8\}$, consider each pair of sets and determine which, if any, is a subset of the other. For example is A a subset of B? B of A? etc..

8. Prove that the set of positive integers divisible by either 2 or 3 is countably infinite.

9. Prove that if A and B are countably infinite sets, then so is $A \cup B$.

10. Prove that if A and B are countably infinite sets, then so is $A \times B$. Hint: describe a dovetailing procedure similar to that used to show that the set of rational numbers is countably infinite.

11. Recall the dovetailing procedure described in Section . In what round will $59/457$ be listed?

12. Recall the dovetailing procedure described in Section . What fraction will be the 1000th listed?

13. List all the subsets of the set $\{1, 2, 3, 4\}$.

14. List all the subsets of $\{\emptyset, \{a\}\}$.

15. How many elements are in $\mathcal{P}(A)$, where A is the set from Exercise 2?

16. Which of the following statements are true?

    a. $\{\{1\}, 1\} \subseteq \{1, \{1\}, \{\{1\}\}\}$.

    b. $\{\{1, \{1\}\}\} \subseteq \{1, \{1\}, \{\{1\}\}\}$.

    c. $\{1, \{1\}\} \in \{1, \{1\}, \{\{1\}\}\}$.

    d. $\{1, \{1\}\} \in \{\{\{1\}\}, \{1, \{1\}\}, 1\}$.

17. Let $A = \{1, 2, 3, 4, 5\}$, and $B = \{0, 3, 6\}$. Find $A \cup B$, $A \cap B$, $A \oplus B$, $A - B$, and $B - A$.

18. Let $A = \{a, b, c, d, e\}$, and $B = \{a, b, c, d, e, f, g, h\}$. Find $A \cup B$, $A \cap B$, $A \oplus B$, $A - B$, and $B - A$.

19. If $A = \{1, 7, 12, 15, 22\}$ and $A \oplus B = \{1, 5, 9, 15, 20, 22, 27\}$, then provide set B.

20. Let $A_i = \{1, 2, \ldots, i\}$, for $i = 1, 2, 3, \ldots$. Use list notation to express

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n,$$

and

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n,$$

21. Repeat the previous problem, but now assume $A_i = \{\ldots, -2, -1, 0, 1, 2, \ldots, i\}$, for $i = 1, 2, 3, \ldots$.

22. Find $A \times B$ and $B \times A$, where $A = \{a, b, c, d\}$, and $B = \{y, z\}$.

23. Provide sets A, B, and C for which $A \times (B \times C) \neq (A \times B) \times C$. Justify your conclusion.

24. Show in general (i.e. not by a specific example) that if A and B are nonempty, and $A \neq B$, then $A \times B \neq B \times A$.

25. Let E denote the set of English words that appear in your textbook, while, P denotes the set of pages. Give an estimate of the size of $E \times P$. Give an estimate of the size of the subset A of $E \times P$, where $(w, p) \in A$ iff word w appears on page p.

26. Use a membership table to prove that the $\oplus$ operation is associative, i.e.

$$A \oplus (B \oplus C) = (A \oplus B) \oplus C.$$

27. What can you say about the sets A and B if $A \oplus B = A$? Hint: use the previous exercise and the fact that $A \oplus A = \emptyset$ for arbitrary set A.

28. Use a membership table to verify that, for any sets A, B, and C, $(B - A) \cup (C - A) = (B \cup C) - A$.

29. Verify De Morgan's first law using a membership table.

30. Use a membership table to prove that

$$A \cup (B \oplus C) = (A \cup B) \oplus (A \cup C)$$

is *not* a set identity.

Exercise Solutions

1. Which of the following pairs of sets are equal?

   a. $\{1, 3, 3, 3, 5, 5, 5, 5, 5\} = \{1, 3, 5\}$ since each set member is only counted once; i.e. both sets are assumed to contain exactly the numbers 1, 3, and 5.

   b. $\{\{1\}\} \neq \{1, \{1\}\}$ since the first set has one member, while the second has two.

   c. $\emptyset \neq \{\emptyset\}$ since the first set has no members, while the second has one, namely the empty set.

2. $\{1, 4, 9, 16, 25, 36, 49, 64, 81, 100\}$

3. $\{0, 1, 2, 3, 4, 5, 6\}$

4. $\emptyset$

5. $\{1/2, 1/4, 1/6, \ldots, 1/1000\}$

6. $\{0, 1, 4, 9, 16, \ldots\}$

7. We have $B \subset A$, $C \subset A$, and $C \subset D$.

8. $\{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, \ldots\}$.

9. Let a_1, a_2, a_3 be a listing of members of A, and b_1, b_2, b_3 be a listing of members of B. Then $a_1, b_1, a_2, b_2, a_3, b_3, \ldots$ is a listing of members of $A \cup B$.

10. Let a_1, a_2, a_3 be a listing of members of A, and b_1, b_2, b_3 be a listing of members of B.

    Round 1. Add (a_1, b_1) to the list.

    Round 2. Add (a_1, b_2), and (a_2, b_1) to the list.

    Round 3. Add (a_1, b_3), (a_2, b_2), and (a_3, b_1) to the list.

    Round $i \geq 4$. Add (a_1, b_i), (a_2, b_{i-1}), (a_3, b_{i-2}), $\ldots$, (a_i, b_1) to the list.

11. In Round 59, 59/1 is listed. Moreover, in 456 additional rounds 59/457 is listed. This makes for a total of $59 + 456 = 515$ rounds.

12. We're looking for the first n for which $1 + 2 + \cdots + (n - 1) + n = n(n + 1)/2 \geq 1000$. This means that $n^2 + n - 2000 \geq 0$. But the equation

$$n^2 + n - 2000 = 0$$

has positive solution

$$n = -0.5 + \sqrt{7999}/2 \approx 44.22.$$

Therefore, our desired n is $n = \lceil 44.22 \rceil = 45$. Moreover, after 44 rounds, $(22)(45) = 990$ fractions have been listed, and so we must list 10 additional fractions in Round 45. Finally, the 10 th fraction listed in round 45 is 10/36.

13. The following subset table lists all the subsets of the set $\{1, 2, 3, 4\}$.

$1 \in S$	$2 \in S$	$3 \in S$	$4 \in S$	Subset S
0	0	0	0	$\emptyset$
0	0	0	1	$\{4\}$
0	0	1	0	$\{3\}$
0	0	1	1	$\{3, 4\}$
0	1	0	0	$\{2\}$
0	1	0	1	$\{2, 4\}$
0	1	1	0	$\{2, 3\}$
0	1	1	1	$\{2, 3, 4\}$
1	0	0	0	$\{1\}$
1	0	0	1	$\{1, 4\}$
1	0	1	0	$\{1, 3\}$
1	0	1	1	$\{1, 3, 4\}$
1	1	0	0	$\{1, 2\}$
1	1	0	1	$\{1, 2, 4\}$
1	1	1	0	$\{1, 2, 3\}$
1	1	1	1	$\{1, 2, 3, 4\}$

14. The subsets of $\{\emptyset, \{a\}\}$ are $\emptyset$, $\{\emptyset\}$, $\{\{a\}\}$, and $\{\emptyset, \{a\}\}$.

15. $|\mathcal{P}(A)| = 2^{10} = 1024$

16. Which of the following statements are true?

    a. $\{\{1\}, 1\} \subseteq \{1, \{1\}, \{\{1\}\}\}$. True: both $\{1\}$ and 1 are members of the right set.

    b. $\{\{1, \{1\}\}\} \subseteq \{1, \{1\}, \{\{1\}\}\}$. False: $\{1, \{1\}\}$ is not a member of the right set.

    c. $\{1, \{1\}\} \in \{1, \{1\}, \{\{1\}\}\}$. False: $\{1, \{1\}\}$ is not a member of the right set.

    d. $\{1, \{1\}\} \in \{\{\{1\}\}, \{1, \{1\}\}, 1\}$. True: $\{1, \{1\}\}$ is a member of the right set.

17. $A \cup B = \{0, 1, 2, 3, 4, 5, 6\}$, $A \cap B = \{3\}$, $A \oplus B = \{0, 1, 2, 4, 5, 6\}$, $A - B = \{1, 2, 4, 5\}$, and $B - A = \{0, 6\}$

18. $A \cup B = B$, $A \cap B = A$, $A \oplus B = \{f, g, h\}$, $A - B = \emptyset$, and $B - A = \{f, g, h\}$.

19. B consists of all members of A that do not appear in $A \oplus B$ (since they are in both A and B), together with those members of $A \oplus B$ that do not appear in A (but they do appear in B). In other words, $B = \{5, 7, 9, 12, 20, 27\}$.

20. We have
$$\bigcup_{i=1}^{n} A_i = \{1, 2, \ldots, n\} \text{ and } \bigcap_{i=1}^{n} A_i = \{1\}.$$

21. We have
$$\bigcup_{i=1}^{n} A_i = \{\ldots, -2, -1, 0, 1, 2, \ldots, n\} \text{ and } \bigcap_{i=1}^{n} A_i = \{\ldots, -2, -1, 0, 1\}.$$

22. $A \times B = \{(a, y), (b, y), (c, y), (d, y), (a, z), (b, z), (c, z), (d, z)\}$,
$B \times A = \{(y, a), (y, b), (y, c), (y, d), (z, a), (z, b), (z, c), (z, d)\}$.

23. Let $A = \{1\}$, $B = \{a\}$, and $C = \{z\}$. Then $(1, (a, z)) \neq ((1, a), z)$, since the first tuple has elements 1 and (a, z), while the second has elements $(1, a)$, and z. Hence, they are two different tuples.

24. Assume A and B are nonempty and $A \neq B$. Let $a \in A$ and $b \in B$ be members of A and B, such that either $a \notin B$ or $b \notin A$ is true. **Case 1.** $a \notin B$. Then $(a, b) \in A \times B$, but $(a, b) \notin B \times A$. **Case 2.** $b \notin A$. Then $(b, a) \in B \times A$, but $(b, a) \notin A \times B$. $\square$

25. The estimate equals $w \times p$, where w is your estimate of the number of words per page of your textbook, and p is the number of pages of your textbook.

26. Since its last two columns are identical, the following membership table establishes $A \oplus (B \oplus C) = (A \oplus B) \oplus C$.

$x \in A$	$x \in B$	$x \in C$	$x \in$ $A \oplus B$	$x \in$ $B \oplus C$	$x \in$ $A \oplus (B \oplus C)$	$x \in$ $(A \oplus B) \oplus C$
0	0	0	0	0	0	0
0	0	1	0	1	1	1
0	1	0	1	1	1	1
0	1	1	1	0	0	0
1	0	0	1	0	1	1
1	0	1	1	1	0	0
1	1	0	0	1	0	0
1	1	1	0	0	1	1

27. We have
$$A \oplus (A \oplus B) = (A \oplus A) \oplus B = \emptyset \oplus B = B,$$
where the first equality makes use of the associativity of $\oplus$. But since $A \oplus B = A$, we also have
$$A \oplus (A \oplus B) = A \oplus A = \emptyset.$$
Therefore, $B = \emptyset$.

28. The following membership table establishes $(B - A) \cup (C - A) = (B \cup C) - A$.

$x \in A$	$x \in B$	$x \in C$	$x \in$ $B - A$	$x \in$ $C - A$	$x \in$ $B \cup C$	$x \in$ $(B \cup C) - A$	$x \in$ $((B - A) \cup (C - A))$
0	0	0	0	0	0	0	0
0	0	1	0	1	1	1	1
0	1	0	1	0	1	1	1
0	1	1	1	1	1	1	1
1	0	0	0	0	0	0	0
1	0	1	0	0	1	0	0
1	1	0	0	0	1	0	0
1	1	1	0	0	1	0	0

29. Since the final two columns are equal, the following membership table establishes $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

$x \in A$	$x \in B$	$x \in \overline{A}$	$x \in \overline{B}$	$x \in (A \cup B)$	$x \in \overline{A \cup B}$	$x \in (\overline{A} \cap \overline{B})$
0	0	1	1	0	1	1
0	1	1	0	1	0	0
1	0	0	1	1	0	0
1	1	0	0	1	0	0

30. The following membership table establishes that

$$A \cup (B \oplus C) = (A \cup B) \oplus (A \cup C)$$

is *not* a set identity column 5 is not identical with column 8.

$x \in A$	$x \in B$	$x \in C$	$x \in$ $B \oplus C$	$x \in$ $A \cup (B \oplus C)$	$x \in$ $A \cup B$	$x \in$ $A \cup C$	$x \in$ $(A \cup B) \oplus (A \cup C)$
0	0	0	0	0	0	0	0
0	0	1	1	1	0	1	1
0	1	0	1	1	1	0	1
0	1	1	0	0	1	1	0
1	0	0	0	1	1	1	0
1	0	1	1	1	1	1	0
1	1	0	1	1	1	1	0
1	1	1	0	1	1	1	0