**CECS 329, Homework Assignment 6, Spring 2025, Dr. Ebert**

**Directions: Please review the `Homework` section on page 6 of the syllabus including a list of all rules and guidelines for writing and submitting solutions.**

**Due Date: Friday, May 9th as a PDF-file upload to the HW6 Canvas dropbox.**

# Problems

1. Do the following.

   (a) Provide the instructions of a URM program that computes the function $f(x) = \lfloor \sqrt{x} \rfloor$. Hint: use the fact that $(n+1)^2 = n^2 + (2n+1)$, meaning that to get the $n+1$ perfect square one only needs to add the $n$ th odd number to the $n$th perfect square. (10 pts)

   (b) For each register used by the program in part a), provide a complete-sentence description of its purpose. (20 pts)

2. Use the algorithm from lecture for encoding a URM program in order to devise a formula for the function $h(a)$ so that

   $$\phi_{h(a)}(x) = ax,$$

   where $a \geq 0$ is any natural number. For example, $\phi_{h(0)}(x) = 0x = 0$ and so $h(0)$ is the Gödel number for the zero function. On the other hand, $\phi_{h(3)}(x) = 3x$ and so $h(3)$ is the Gödel number for a program that computes the function $f(x) = 3x$. Hint: examine the LO9b solution from the April 24th assessment and both modify and generalize it. Make sure to explain how you obtained your solution. Otherwise, points will be lost. Also, simplify the arithmetic expression for $h(x)$ as much as possible. (20 pts)

3. Answer the following. Note: adequately solving each part of this problem counts for passing LO12.

   (a) Given some URM-computable function $f(x, y)$, the proof of Kleene's Theorem describes a three-part program $P_e = ABC$ for which, for all $y \in \mathbb{N}$, $P_e(y) = f(e, y)$. Provide two of its most important responsibilities for parts $A$ and $B$, and one for part $C$. (10 pts)

   (b) An instance of the `Different LSD` decision problem is a Gödel number $x$, and the problem is to decide if for *all* inputs $y \in \mathbb{N}$, either $P_x(y) = \uparrow$ *or* $P_x(y) = z$ and the least significant digit (LSD) of $y$ is different from that of $z$. For example, the numbers 35 and 145 have the same LSD of 5, where as the numbers 13 and 17 have the different LSD's 3 and 7. For each of the following three Gödel numbers $e_1$, $e_2$, and $e_3$ determine which one's are positive instances of `Different LSD`. Defend your answers. (5 pts each)

      i. $e_1$ is the Gödel number for which $\phi_{e_1}(x) = 10x + 3$.

      ii. $e_2$ is the Gödel number for which $\phi_{e_2}(x) = x^2 + 7$.

iii. $e_3$ is the Gödel number for which $\phi_{e_3}(x) = d_{\text{lsd}}(x)$, where $d_{\text{lsd}}(x)$ is the decision function for the `Different LSD` decision problem (assuming that the problem is decidable and hence that $d_{\text{lsd}}(x)$ is total computable).

(c) Prove that the `Different LSD` decision problem is undecidable. Do this by assuming that the decision function $d_{\text{lsd}}(x)$ for `Different LSD` is total computable and provide a program $P$ that makes use of i) $d_{\text{lsd}}(x)$ and ii) the `self` programming construct to create a contradiction. Provide a case analysis that shows why $P$ does indeed create a contradiction. Conclude that $d_{\text{lsd}}(x)$ must not be computable, and hence `Different LSD` is an undecidable problem. (15 pts)