CECS 528, Homework Assignment 7, Spring 2025, Dr. Ebert

Directions: Please review the `Homework` section on page 7 of the syllabus including a list of all rules and guidelines for writing and submitting solutions.

Due Date: Friday May 9th as a PDF-file upload to the HW7 Canvas dropbox.

Problems

1. Consider the following description of a randomized algorithm, call it $\mathcal{K}$, for finding the minimum-cut for a simple graph $G = (V = \{v_1, \ldots, v_n\}, E)$. Start with a forest of $n = |V|$ trees where the i th tree consists of the single vertex v_i. Then while there are more than two trees in the forest, randomly select an edge $e = (u, v)$ which, when added to the forest, does not create a cycle. Then u and v must lie in different trees T_u and T_v and these are merged into the single tree $T_{u \cup v}$.

   (a) Explain why this algorithm is exactly the same as Karger's algorithm. In other words, if the same sequence of edges were randomly selected by both algorithms, then both would return the exact same cut set. In particular, when two vertices are merged in a step of Karger's algorithm, what does this correspond with in algorithm $\mathcal{K}$? Explain why both algorithms cause the exact same edges to be removed, whether it be by Karger's merger step or by the equivalent operation performed in the corresponding step of algorithm $\mathcal{K}$. (15 pts)

   (b) Show the steps of both algorithms (side by side by dividing your paper length-wise into two columns) when applied to the graph whose edges are

   $$\{e_1 = (1, 4), e_2 = (4, 6), e_3 = (1, 2), e_4 = (2, 5), e_5 = (2, 3),$$

   $$e_6 = (3, 5), e_7 = (3, 6), e_8 = (5, 6), e_9 = (1, 3)\},$$

   and for which e_6, e_9, e_4, and e_7 have been randomly selected in rounds 1, 2, 3, and 4, respectively. (10 pts)

2. Consider the problem of applying `WalkSAT` to a satisfiable instance of `3SAT` that has four variables and a unique satisfying assignment β. Letting F be the random variable that measures the number of bit flips that leads up to the event $\alpha = \beta$, our goal is to compute an upper-bound for $E[F|D = i]$, $i = 0, 1, \ldots, 4$, where D is a random variable that represents the current Hamming distance $d(\alpha, \beta)$. Determine upper bounds for the values $E[F|D = 0], \ldots, E[F|D = 4]$, using both recurrences and edge cases that are inspired by the recurrence equation provided in item 6 of the proof outline of Theorem 5.1 of the Randomized Algorithms lecture. As an example, a good upper bound in case $i = 3$ is

   $$E[F|D = 3] = \frac{2}{3}(1 + E[F|D = 2]) + \frac{1}{3}(1 + E[F|D = 4]).$$

Here $\frac{2}{3}$ is chosen by considering the following worst-case scneario. Suppose c is a randomly selected unsatisfied clause. Since $D = 3$, at most one of the literals of c is correctly assigned (why?). Thus if a literal of c is randomly and uniformly selected, then with probability $2/3$ flipping the assignment bit for this literal will result in one additonal literal being in agreement with β, i.e. reducing the Hamming distance from 3 to 2. Use similar analyses for the cases $D = 1, 2, 4$ along with the edge case for $D = 0$ to obtain five equations with respect to five variables and solve the system of equations. Note: correctly solving this problem counts for passing LO12. (20 pts)

3. Let G be an unweighted simple graph having m edges, and having a max cut equal to cm, for some $c \geq 0.844\ldots$. Show that the Goemans-Williamson algorithm produces a cut whose expected weight is at least αcm, where

$$\alpha = \frac{\cos^{-1}(1 - 2c)}{\pi c}.$$

Which value of c wouuld maximize α? Show work. Note: correctly solving this problem counts for passing LO12. (25 pts)