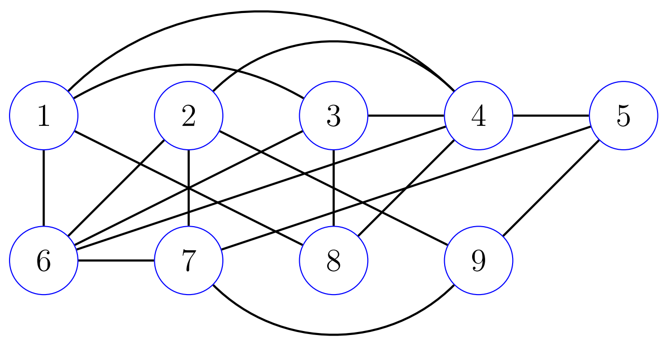
CECS 528, Learning Outcome Assessment 9, 4/16/2025, Dr. Ebert

Problem

LO6. Do the following.

(a) Provide the definition of what it means to be a mapping reduction from problem A to problem B.

(b) The simple graph $G = (V, E)$ shown below along with $k = 4$ is an instance of the `Clique` decision problem. Draw $f(G, k)$, where f is the mapping reduction from `Clique` to `Half Clique` provided in one of the lecture exercises.



(c) Verify that f is valid for input (G, k) from part b in the sense that both (G, k) and $f(G, k)$ are either both positive instances or both negative instances. Make sure your answer is specific to the particular instances from part b.

LO7. An instance of the `Max Cut` decision problem is a weighted simple graph $G = (V, E, w)$ and an integer $k \geq 0$. The problem is to decide if there is a subset $U \subseteq V$ for which

$$w(e_1) + w(e_2) + \cdots + w(e_r) \geq k,$$

where $\{e_1, e_2, \ldots, e_r\}$ is the set of all edges e for which one vertex of e is a member of U, and the other vertex is a member of $E - U$.

(a) For a given instance (G, k) of `Max Cut`, describe a certificate in relation to (G, k).

U is a subset of vertices : $U \subseteq V$.

(b) Provide a semi-formal verifier algorithm that takes as input i) an instance (G, k) of `Max Cut`, ii) a certificate for (G, k) as defined in part a, and decides if the certificate is valid for (G, k).

Sum = 0.
For each $e = (u, v) \in E$,
If $(u \in U \wedge v \in V - U) \vee (v \in U \wedge u \in V - U)$ sum += $w(e)$.
If sum $\geq$ K, return 1.
Return 0.

(c) Assuming that $w : E \to \mathcal{N}$ maps edges to natural numbers, clearly define three size parameters for the `Max Cut` decision problem.

$b = \lfloor \log(K) \rfloor + 1$
$m = |E|$ $n = |V|$

(d) Use the size paramters from part c to describe the running time of your verifier. Defend your answer.

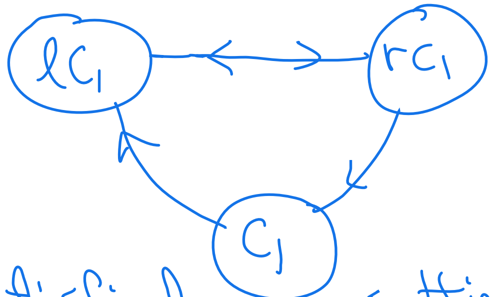
$O(m)$ steps to build hash tables for storing U and $V - U$ to facilitate the membership queries that occur in each iteration. $O(m)$ iterations of the loop with $O(b)$ steps per iteration. ∴ $O(mb)$ number of steps.

LO8. Answer the following.

(a) Consider the mapping reduction $f : \texttt{3SAT} \to \texttt{DHP}$ from 3SAT to DHP that was presented in lecture and the 3SAT instance

$$\mathcal{C} = \{c_1 = (x_1, x_2, x_3), c_2 = (\overline{x}_2, x_3, \overline{x}_4), c_3 = (x_1, x_2, \overline{x}_4), c_4 = (\overline{x}_1, \overline{x}_3, \overline{x}_4), c_5 = (\overline{x}_1, x_2, x_4)\}.$$

i. Draw the subgraph of $f(\mathcal{C})$ whose vertices are lc_1, rc_1, and c_1, where lc_1 and rc_1 belong to the chain of vertices in the x_1-diamond and c_1 is the clause-1 vertex. Explain your drawing.



Since c_1 is satisfied by setting $x_1 = 1$, this corresponds with moving right-to-left in diamond which is the only way to visit c_1 from x_1-diamond without corrupting the path from a to b.

ii. Which diamond of $f(\mathcal{C})$ does not have edges connecting to clause vertex c_4? Explain.

Diamond x_2 since neither x_2 nor $\overline{x}_2$ satisfies c_4.

iii. Is there a Hamilton path from a to b in $f(\mathcal{C})$ that moves from left to right through each of the diamonds of $f(\mathcal{C})$? Explain

If so, it would imply that $\alpha = (x_1 = x_2 = x_3 = x_4 = 0)$ satisfies $\mathcal{C}$. But α does not satisfy c_1.

(b) Consider the mapping reduction $f : \texttt{3SAT} \to \texttt{Clique}$ from 3SAT to Clique that was presented in lecture. Let $\mathcal{C}$ be the 3SAT instance from part a) of this LO. How many vertices and edges does the graph of $f(\mathcal{C})$ have? Show work.

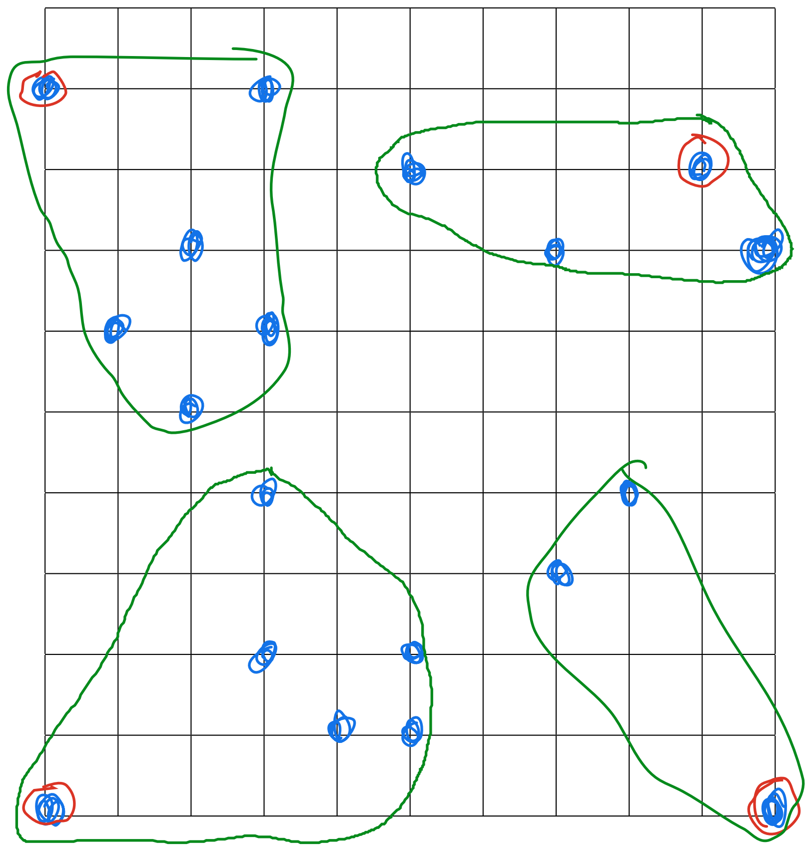
$m = 5$. Thus, $f(\mathcal{C})$ has $(3)(5) = \boxed{15}$ vertices and $\binom{5}{2} \cdot 9 - (4 + 3 + 2 + 3) = 78$ edges

LO9. Do the following.

(a) Draw the set

$$S = \{(0,0), (0,9)(1,6), (2,5), (2,7), (3,2), (3,4), (3,6), (3,9), (4,1), (5,1),$$
$$(5,2), (5,8), (7,3), (7,7), (8,4), (9,8), (10,0), (10,7)\}$$

of points and apply the `k-Clustering` algorithm to S and $k = 4$. Select $(0,9)$ as the first center. Clearly indicate which points are cluster centers and enclose each cluster with a boundary curve. Label each cluster (e.g. C_1) in accordance with the order in which it's center was selected. Note: all distances are Euclidean: $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.



(b) Answer the following regarding the anaysis of Christofides' approximation algorithm for TSP, assuming that edge distances satisfy the triangle inequality.

i. Christofides' algorithm requires performing an Euler tour through some graph. What are the two sources of edges for this graph?

The edges are from the mst T of the graph G and the edges of G that yield the min-cost matching for the subgraph H whose vertices are the odd-degree vertices of T.

ii. Given an optimal TSP Hamilton cycle $C = b, f, d, c, e, i, a, h, l, g, b$, provide a corresponding Hamilton cycle C' for the complete subgraph whose vertices are $\{e, f, g, h\}$ and for which $\text{cost}(C') \leq \text{cost}(C)$. Why is this inequality true for your provided cycle?

$C' = e, h, g, f, e$. $\text{cost}(C') \leq \text{cost}(C)$ since, e.g., moving from e directly to h costs no more than moving from e to h via e, i, a, h. This is due to the Δ-inequality.

iii. Christofides' algorithm provides an approximation ratio of 1.5. Use part ii) to explain how the 0.5 is obtained.

Since C has optimal cost C' has cost that is $\leq$ optimal. Moreover, C' consists of matchings

$M_1 = \{(e,h), (g,f)\}$ and $M_2 = \{(h,g), (f,e)\}$

Thus, assuming $\{e,f,g,h\}$ are the odd-degree vertices of the mst, we have that $\text{cost}(M_1)$ or $\text{cost}(M_2) \leq 0.5\,\text{cost}(C)$ and thus the min-cost matching M must also satisfy $\text{cost}(M) \leq 0.5\,\text{cost}(C)$. Of course, the mst edges also have a cost that is $\leq \text{cost}(C)$. Hence, both together have total cost $\leq 1.5\,\text{cost}(C)$

LO10. An urn consists of six balls: 2 reds, 2 blues, and 2 greens. Suppose two balls are randomly selected from the urn. Let R, B, and G denote the number of reds, blues, and greens that were selected, respectively.

$\binom{6}{2} = 15$

(a) Provide the domain and probability distribution for each of these random variables. Show work.

$\text{dom}(R) = \text{dom}(B) = \text{dom}(G) = \{0, 1, 2\}$

$P(0) = \frac{6}{15} = \frac{2}{5}$, $P(1) = \frac{8}{15}$, $P(2) = \frac{1}{15}$

Note: $P(0) + P(1) + P(2) = 1$.

This distribution applies to all three variables.

(b) Compute both $E[R]$ and $E[R|B = 1]$. Note: all answers should be in the form of fractions.

$E[R] = \frac{8}{15} + (2)\left(\frac{1}{15}\right) = \frac{10}{15} = \frac{2}{3}$

0

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$E[R \mid B=1] = P(R=1 \mid B=1) + 2P(R=2 \mid B=1)$

$= P(R=1 \mid B=1) = \frac{1}{2}$. Note $\binom{6}{2} = 15$ is the number of different pairs that can be selected.

(c) Suppose after selecting the balls we keep them if they are of the same color, or place them back in the urn and randomly re-select. This process continues until we have selected two balls of the same color. How many times should we expect to have to select from the urn before selecting two balls of the same color? Show work and explain.

Selecting two balls of the same color will occur with probability $\frac{3}{15} = \frac{1}{5}$. ∴ Since the # of attempts follows a geometric distribution with $p = \frac{1}{5}$, it follows that 5 attempts are expected.