CECS 528, Learning Outcome Assessment 9, 4/16/2025, Dr. Ebert

Problem

LO5. Consider the `2SAT` instance

$$\mathcal{C} = \{(x_1, \overline{x}_2), (x_1, \overline{x}_5), (\overline{x}_1, x_3), (\overline{x}_1, \overline{x}_6), (\overline{x}_2, x_5), (x_3, \overline{x}_4), (\overline{x}_3, \overline{x}_4), (x_4, \overline{x}_6)\}.$$

(a) Draw the implication graph $G_{\mathcal{C}}$.

(b) Perform the `Improved 2SAT` algorithm by computing the necessary reachability sets. Use numerical order (in terms of the variable index) and positive literal before negative literal when choosing the reachability set to compute next. Draw the resulting reduced `2SAT` instance whenever a consistent reachability set is computed. Either provide a final satisfying assignment for $\mathcal{C}$ or indicate why $\mathcal{C}$ is unsatisfiable.

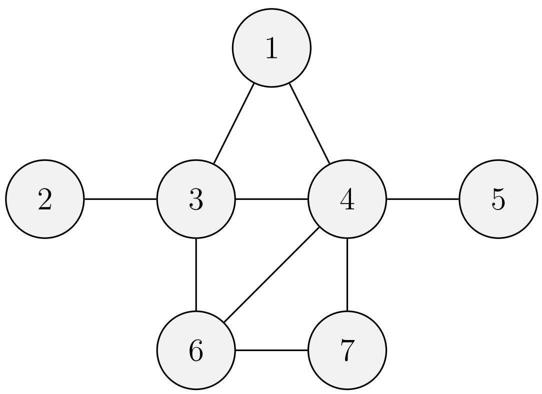
(c) Suppose instance $\mathcal{C}$ of `2SAT` is unsatisfiable. What is the least possible number of queries to the `Reachability`-oracle that you would need in order to confirm $\mathcal{C}$ being unsatisfiable? Explain.

LO6. Answer the following.

(a) Provide the definition of what it means to be a mapping reduction from problem A to problem B.

See Lecture Notes

(b) The simple graph $G = (V, E)$ shown below is an instance of the `Hamilton Path (HP)` decision problem. Provide $f(G)$, where $f : \texttt{HP} \to \texttt{LPath}$ is the mapping reduction from `HP` to `LPath` described in lecture.



$f(G) = (G, K = \underset{|V|-1}{\underset{\shortparallel}{6}})$

(c) Verify that f is valid for input G from part b in the sense that both G and $f(G)$ are either both positive instances or both negative instances. Make sure your answer is specific to the particular instances from part b.

Any HP would have to start/end at 2 and end/start at 5 (why?). When starting at 2, the next vertex visited is 3. If the path starts as 2, 3, 1, 4, then no HP exists, since there is no way to access 5 once 6 and 7 are visited since the path cannot return to 4. The same is true for paths that begin with 2, 3, 6, 7, 4 and 2, 3, 6, 4. Thus, G has no simple paths of length 6.

LO7. An instance of the `Matrix Cover` decision problem is an $n \times n$ 0-1 matrix A and a nonnegative integer k. A **component** of A is either a row or column of A. Thus, A has $2n$ different components (n rows and n columns). The problem is to decide if A has k components for which every 1 entry of A belongs to at least one of the components.

(a) For a given instance (A, k) of `Matrix Cover`, describe a certificate in relation to (A, k).

A certificate consists of two n-bit Boolean arrays c and r, where $\sum_{i=1}^{n} (c[i] + r[i]) = k$.

(b) Provide a semi-formal verifier algorithm that takes as input i) an instance (A, k) of `Matrix Cover`, ii) a certificate for (A, k) as defined in part a, and decides if the certificate is valid for (A, k).

For each $i, j \in \{1, \ldots, n\}$
  If $A[i,j] = 1$ and $r[i] = 0$ and $c[j] = 0$,
    then Return 0. // Entry (i,j) is not covered
Return 1.

(c) Provide the size parameter for `Matrix Cover` and use it to provide a big-O bound on the verifier's running time. Justify your answer.

n = matrix dimension

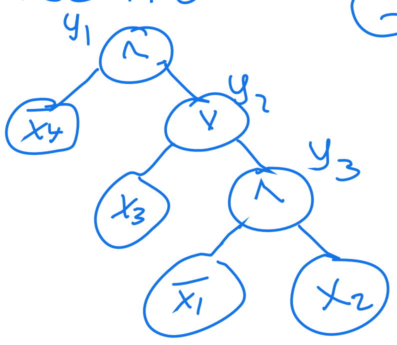
Running time is $O(n^2)$ since the loop requires n^2 iterations and the work performed for each iteration consists of $O(1)$ matrix and array accesses.

LO8. Answer the following.

(a) Recall the Tseytin transformation $f : \texttt{SAT} \to \texttt{3SAT}$ that is used to map an instance F of $\texttt{SAT}$ to an instance $f(F)$ of $\texttt{3SAT}$.

i. If $F(x_1, x_2, x_3, x_4) = \overline{x}_4 \wedge (x_3 \vee (\overline{x}_1 \wedge x_2))$, then draw its parse tree and provide the formula G that is satisfiability equivalent to F. Hint: G is the initial formula that makes use of the y-variables.

Parse Tree



$$G(x_1, x_2, x_3, x_4, y_1, y_2, y_3) = y_1 \wedge (y_1 \leftrightarrow (\overline{x}_4 \wedge y_2)) \wedge (y_2 \leftrightarrow (x_3 \vee y_3)) \wedge (y_3 \leftrightarrow (\overline{x}_1 \wedge x_2))$$

ii. Provide an instance of $\texttt{3SAT}$ that is logically equivalent to the Boolean formula

$$\overline{x_3 \vee \overline{y}_2} \to x_5. \Leftrightarrow (\overline{x}_3 \wedge y_2) \to x_5$$

$$\Leftrightarrow \overline{(\overline{x}_3 \wedge y_2)} \vee x_5 \Leftrightarrow x_3 \vee \overline{y}_2 \vee x_5$$

$$\Rightarrow \mathcal{C} = \{(x_3, \overline{y}_2, x_5)\}$$ has one clause.

(b) Consider the mapping reduction $f : \texttt{3SAT} \to \texttt{DHP}$ from $\texttt{3SAT}$ to $\texttt{DHP}$ that was presented in lecture. Given the $\texttt{3SAT}$ instance

$$\mathcal{C} = \{c_1 = (x_1, x_2, x_3), c_2 = (\overline{x}_2, x_3, \overline{x}_4), c_3 = (x_1, x_2, \overline{x}_4), c_4 = (\overline{x}_1, \overline{x}_3, \overline{x}_4), c_5 = (\overline{x}_1, x_2, x_4),$$

$$c_6 = (\overline{x}_2, x_3, x_4), c_7 = (x_1, \overline{x}_3, x_4), c_8 = (\overline{x}_2, \overline{x}_3, x_4), c_9 = (\overline{x}_2, \overline{x}_3, \overline{x}_4)\},$$

is the following itinerary valid for producing a DHP in $f(\mathcal{C})$? Itinerary: i) move right to left through the x_1-diamond while visiting clauses c_1, c_3, c_7, ii) move left to right through the x_2-diamond while visiting clauses c_2, c_6, c_8, c_9, iii) move left to right through the x_3-diamond while visiting clause c_4, iv) move right to left through the x_4-diamond while visiting clause c_5. This itinerary is valid since

i) the movement directions correspond with the assignment $\alpha = (x_1 = 1, x_2 = 0, x_3 = 0, x_4 = 1)$ which satisfies $\mathcal{C}$.

ii) Each clause is visited, and is visited from a diamond using a direction that satisfies the clause when assigning the diamond variable the direction

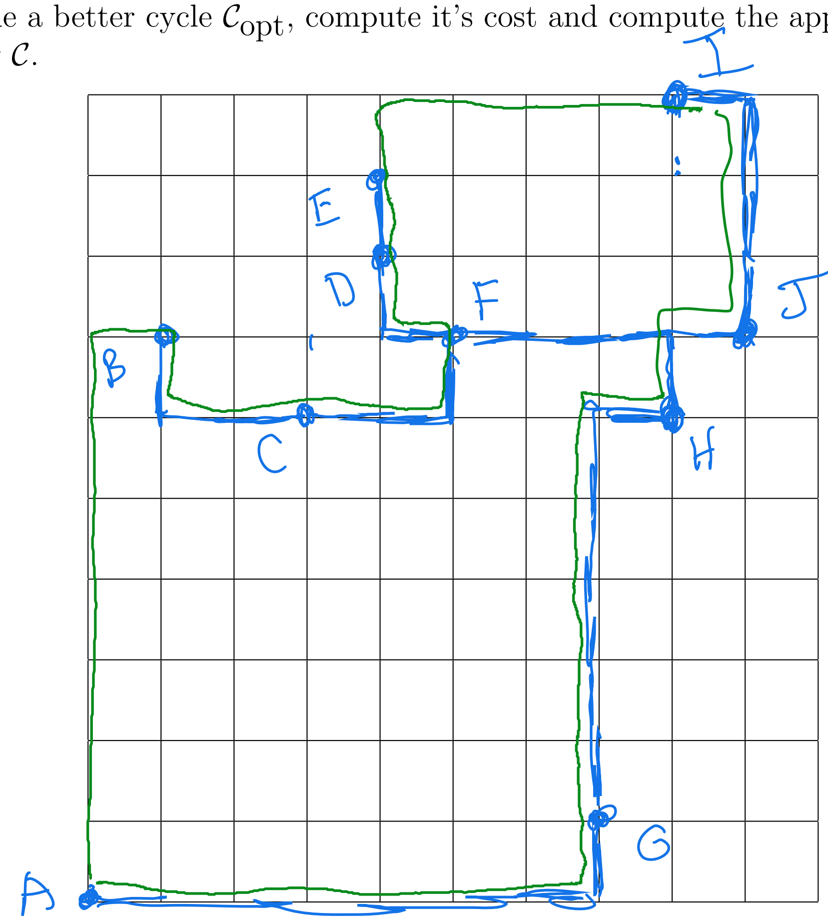
e.g. 1 = right to left
0 = left to right.

LO9. Answer the following.

(a) A delivery truck starts at intersection $A = (0, 0)$, and must deliver packages to intersections

$$B = (1, 7), C = (3, 6), D = (4, 8), E = (4, 9), F = (5, 7), G = (7, 1), H = (8, 6), I = (8, 10)$$
$$\text{and } J = (9, 7).$$

i) Plot the a grid as well as the mst, ii) Assuming that A is the initial vertex, provide the approximation Hamilton Cycle $\mathcal{C}$ and compute its cost, iii) Is $\mathcal{C}$ optimal? If not, then provide a better cycle $\mathcal{C}_{\text{opt}}$, compute it's cost and compute the approximation ratio attained by $\mathcal{C}$.



Cost($\mathcal{C}_{opt}$) = 42
Ratio = $\frac{44}{42} = \frac{22}{21} < 2$

$\mathcal{C}$ = A, G, H, J, I, F, D, E, C, B, A

cost($\mathcal{C}$) = 44. $\mathcal{C}$ is not optimal. $\mathcal{C}_{opt}$ = A, G, H, J, I, E, D, F, C, B, A

iii) For any optimal solution, x must be in a cluster that has at least one center. Thus this cluster has diameter ≥ r.

Contains y and z. Since $d(x,c), d(y,c) \le r$, $d(x,y) \le r + r = 2r$.

(b) In the analysis of the k-`Clustering` approximation algorithm recall that point x is introduced as what would have been the next center (assuming $k + 1$ or more desired clusters). i) How is the value r defined in relation to x? ii) How do we know that any two points y and z in some cluster are within a distance of $2r$ from each other? iii) How do we know that the optimal solution must have at least one cluster with a diameter of at least r?

i) $r = d(x, C)$ where C is the set of centers.

ii) Let c denote the center of the cluster that