Finite Automata

Last Updated: October 6th, 2025

1 Languages

Sets of words over some alphabet play a central role in computer science.

Definition 1.1. An **alphabet** $\Sigma = \{s_1, \ldots, s_n\}$ is a set of symbols. A **word** w over Σ is a sequence of zero or more symbols belonging to Σ. Σ^* denotes the set of all possible words over Σ, including the empty word ε. The **length** of a word w, denoted $|w|$, equals the length of the sequence of symbols that comprises w.

$\Sigma = \{a, b, c, \ldots, z\}$
$cat \in \Sigma^*$ $\quad |cat| = 3$

Example 1.2. If $\Sigma = \{a, b\}$, then aaa, abaabba and bbab are all words over Σ, and $|\text{abaabba}| = 7$. Also $|\varepsilon| = 0$. The expression $\{a, b\}^*$ denotes all words over the alphabet $\{a, b\}$.

The most common operation on words is concatentation. Given words v and w in Σ^*, $v \cdot w$ is the **concatentation** of v with w, and equals the word whose i th letter is v_i if $1 \leq i \leq |v|$, and $w_{i-|v|}$ if $|v| < i \leq |v| + |w|$. Like arithmetic multiplication, we sometimes omit the dot and simply write vw for the concatentation of v with w.

$v = dog$ $\quad v \cdot w = doghouse$
$w = house$ $\quad = vw$

Definition 1.3. A subset of words over some alphabet Σ is called a **language** over Σ.

Example 1.4. The `Prime` language, is the set of binary words w for which there is a prime number n for which $(n)_2 = w$. The words in `Prime` of length four or less are

$$10, 11, 101, 111, 1011, \text{ and } 1101.$$

$(2)_2$ $(7)_2$ $(13)_2$

In addition to the above definition of language, the notion of a **formal language** is also central in computer science. Informally, a language is considered formal if there is a set of rules that, when followed, produce words in the language. Such languages can be used to define programming languages, communication protocols, and search for data in files and databases. We will see that for each type of computing machine we define, there is a language that is naturally associated with the machine, namely the language that encodes the decision problem that is decided by the machine. For deterministic and non-deterministic finite automata, we call these languages **regular**, for pushdown automata we call them **context free**, and for Turing machines we call them **recursive** or **recursively enumerable**.

$(x_1 \vee \bar{x}_2) \wedge x_3$ $\Sigma = \{(,), \vee, \wedge, x_1, \bar{x}_1, x_2, \bar{x}_2, \ldots\}$

2 Introduction

The finite automaton represents one of the most basic models of computation. A finite automaton consists of a finite number of states which serve as the automaton's only source of memory. Since an automaton is capable of accepting languages having arbitrarily long words, it must read each input word (i.e. problem instance) symbol by symbol until all symbols have been read at which point it either accepts or rejects w.

Finite automata have applications in several areas of computing, including computer architecture, cellular automata, natural language processing and recognition, compiler theory, and text processing to name just a few. Finite automata solve problems that require only a finite amount of memory, regardless of the problem size.

A **deterministic finite automaton (DFA)** consists of a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

Q is a finite set with each member of Q called a **state**

Σ a finite alphabet that is used to represent an input word

δ a transition function that determines the next state of the automaton given the current state and the current input symbol being read. In other words,

$$\delta : Q \times \Sigma \to Q,$$

is a mapping from (current) state-input pairs to (next) states.

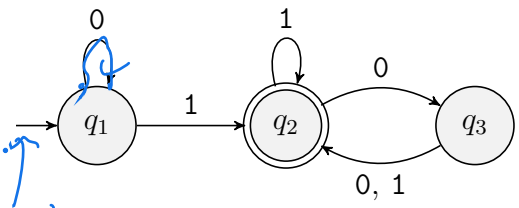
$q_0 \in Q$ the **initial state**.

$F \subseteq Q$ a member of F is called an **accepting state**.

Note that a DFA is capable of reading each input symbol only once, and in the order in which it appears in the input word w.

A graphical way of representing a DFA is to represent the states with graph nodes, with an arrow pointing at the initial state, and accepting states circled. Furthermore, a directed edge labeled with symbol s starts at node q_i and terminates at node q_j iff $\delta(q_i, s) = q_j$.

Example 2.1. Provide the 5-tuple for the following DFA.



points to initial state

$Q = \{q_1, q_2, q_3\}$

$\Sigma = \{0, 1\}$

δ	0	1
q_1	q_1	q_2
q_2	q_3	q_2
q_3	q_2	q_2

q_1 : initial state

Accepting states: $F = \{q_2\}$

2.1 DFA computation

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA, and $w = w_1 w_2 \cdots w_n$ be a word in Σ^*. Then the **computation of M on input** w is a sequence $q_0, q_1, q_2, \ldots, q_n$ of states recursively defined by

w - word
w_i - letter of w

1. q_0 is the initial state of M; and
2. $q_i = \delta(q_{i-1}, w_i)$, for every $1 \leq i \leq n$.

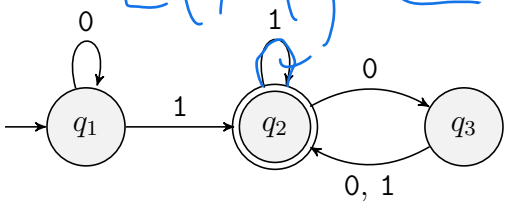
current state
current read symbol/letter

In words, the computation of M on input w is the sequence of states $q_0, q_1, q_2, \ldots, q_n$ that M moves through when successively reading input symbols $w_1, w_2, \ldots, w_n$. The computation of M on input w is said to be **accepting** iff $q_n \in F$. In this case we say M **accepts** w. Otherwise it is called a **rejecting** computation, and M **rejects** w.

Note: a DFA-computation simulator can be found at

`https://www.cs.cmu.edu/~cburch/survey/dfa/index.html`

Example 2.2. For the DFA M shown below, provide the computation of M on input i) $w_1 = 01101011$ and input ii) $w_2 = 0011000$.



$L(M) =$ At least one 1 and either end with a 1 or end with an even # of 0's

final state

$w_1 =$ 0 1 1 0 1 0 1 1

q_1 q_1 q_2 q_2 q_3 q_2 q_3 q_2 q_2

computation of M on input w_1

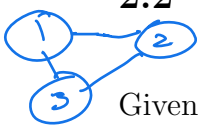
accept since $q_2 \in F$

$w_2 =$ 0 0 1 1 0 0 0

q_1 q_1 q_1 q_2 q_2 q_3 q_2 q_3

reject since $q_3 \notin F$

2.2 Regular language



Given DFA M, the **language accepted by** M is defined as

$$L(M) = \{w \in \Sigma^* | M \text{ accepts } w\}.$$

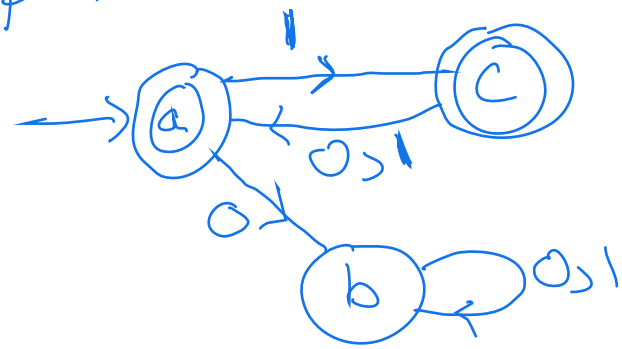
If a language $L \subseteq \Sigma^*$ is accepted by some DFA M, then L is called a **regular language**.

Example 2.3. Let L be the set of binary words w such that every odd bit of w equals 1. Show that L is a regular language.

a: read odd input
b: reject input
c: read even input

$w_1 = 11101 \in L$
1 2 3 4 5

$w_2 = 1001 \notin L$
1 2 3



w_2: 1 0 0 1
a c a b b
⇓
reject

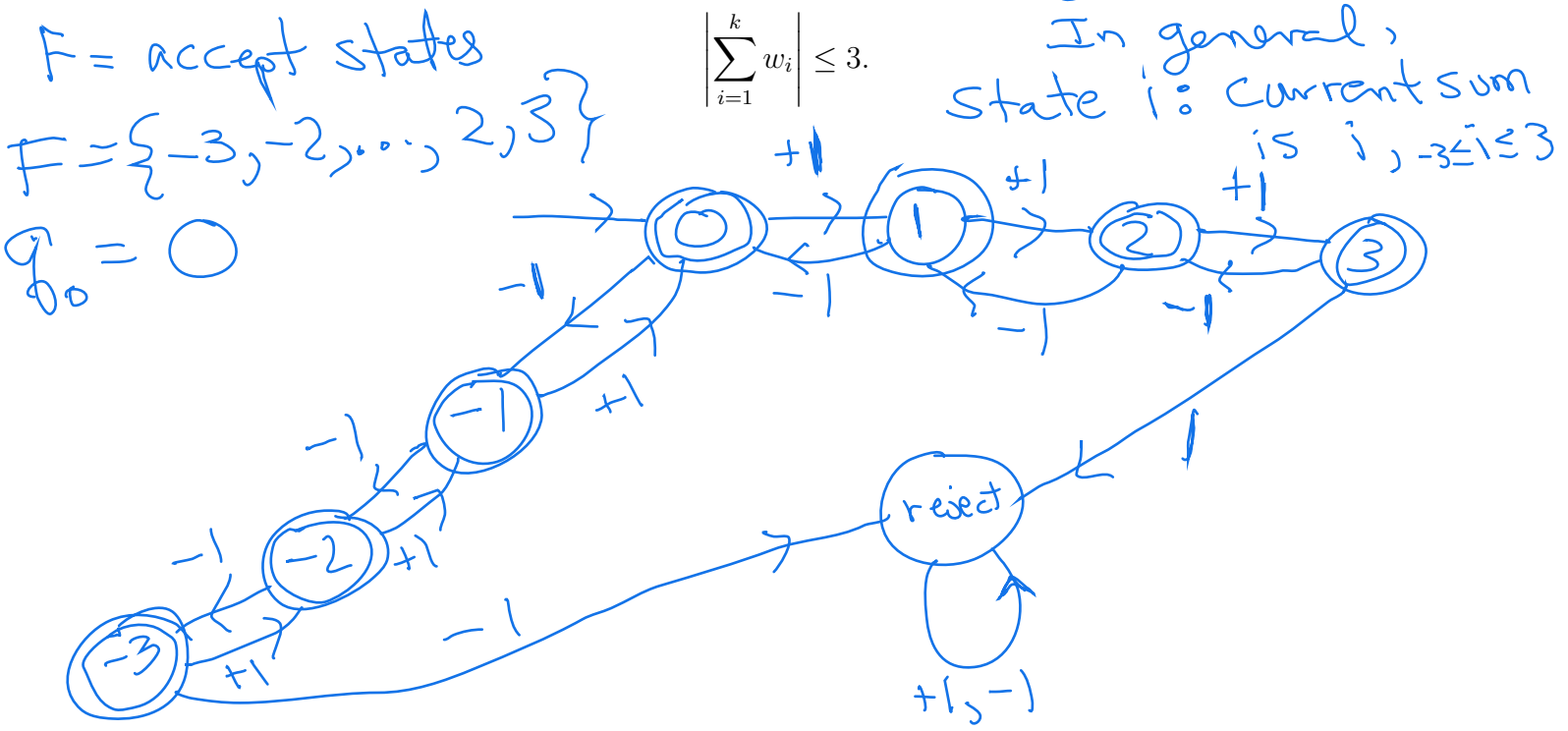
w_1: 1 1 1 0 1
a c a c a c ⇒ accept

Example 2.4. Provide a succinct description of the language that is accepted by the DFA shown in Example 1.

See Top of Example 2.2

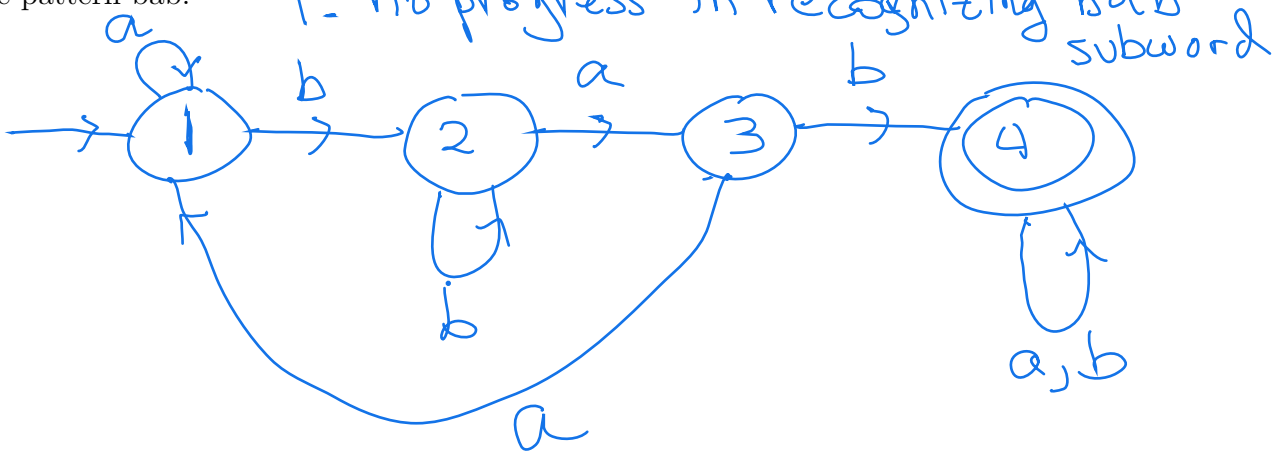
Example 2.5. Provide the state diagram for a DFA that accepts the language of all words w over the alphabet $\{-1, +1\}$ for which, for all $k = 1, \ldots, |w|$,

$$\left| \sum_{i=1}^{k} w_i \right| \leq 3.$$



2: read b 3: read ba 4: read bab

Example 2.6. Provide the state diagram for a DFA that accepts all words over the alphabet {a,b} that contain the pattern bab.



Example 2.7. Consider the alphabet

$$\Sigma = \left\{ \begin{matrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{matrix} \right\}$$

where each symbol is a triple of bits.

1. Provide the δ-transition table for a DFA that accepts all words w over Σ for which the top layer of w minus the middle layer of w equals the bottom layer of w, where each layer is viewed as a binary number whose left most bit is the least significant bit, and whose rightmost bit is the most significant bit. For example, consider the two words

$$w_1 = \begin{matrix} 0001 \\ 1010 \\ 1100 \end{matrix}, \quad \text{and} \quad w_2 = \begin{matrix} 11001 \\ 10100 \\ 01010 \end{matrix}.$$

The DFA should accept w_1 since the top layer represents the number 8, the middle layer 5, the bottom layer 3, and $8 - 5 = 3$. However, it should not accept w_2 since the top layer represents the number 19, the middle layer 5, the bottom layer 25, and $19 - 5 = 14 \neq 10$.

NB	NB	R	R	B	R	NB	NB	R
B	R	B	B	R	NB	R	R	B

2. Show the computation of the DFA for input w_1 defined above.

The proof of the following theorem is left as an exercise.

Theorem 2.8. Let `Regular` denote the class of all regular languages. Then `Regular` $\subseteq$ `P`

A Hierarchy of Languages

Regular computed with finite automata

Context Free computed with pushdown automata (nondeterministic finite automaton with additional stack memory)

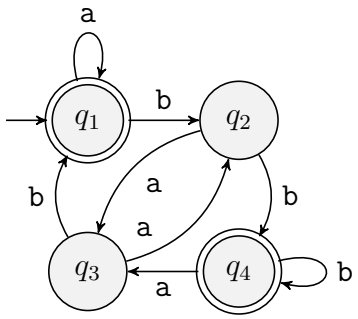
Context Sensitive computed with a Turing machine that uses a linear amount of memory with respect to the size of the input.

Recursive computed with a Turing machine that halts on all inputs.

Recursively Enumerable computed with a Turing machine that halts on all positive instances.

DFA Core Exercises

1. Consider the following state diagram for a DFA M.



   Provide the following for M.

   (a) start state
   (b) set of accept states
   (c) the sequence of states that M goes through on input `aabb`
   (d) Does M accept `aabb`? ε?

2. Provide a formal definition for the DFA M from the previous exercise.

3. Consider the formal DFA description

$$M = (\{q_1, q_2, q_3, q_4, q_5\}, \{u, d\}, \delta, q_3, \{q_3\}),$$

   where δ is given by the following table.

$q_i \backslash s$	u	d
q_1	q_1	q_2
q_2	q_1	q_3
q_3	q_2	q_4
q_4	q_3	q_5
q_5	q_4	q_5

   Draw the state diagram for this DFA.

4. For the machine M provided in the previous exercise, provide the computation of M on input ududduddu. Is this input accepted or rejected?

5. Provide the state diagram for a DFA that accepts all words from $\{a, b\}^*$ that have an even number of a's.

6. Repeat the previous exercise, but now assume the DFA accepts all words from $\{a, b\}^*$ that have an odd number of a's and end with a b.

7. Provide a DFA that accepts all binary words that contain the substring 0101.

8. Provide a DFA that accepts all binary words that do *not* contain 0101.

9. Provide a DFA that only accepts ε and 10.

10. Provide a DFA that accepts all words over the alphabet {a,b} that have an even number of a's or (inclusive) an odd number of b's.

Solutions to DFA Core Exercises

1. We have the following.

   (a) q_1

   (b) $F = \{q_1, q_4\}$

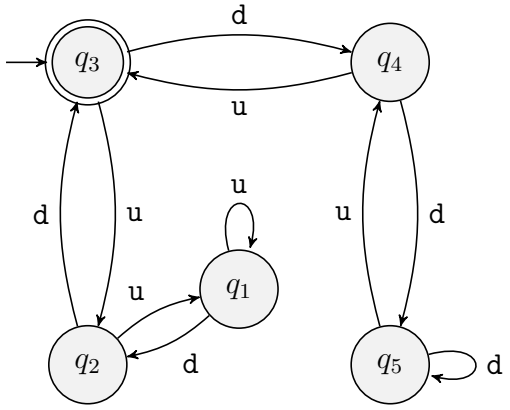
   (c) q_1, q_1, q_1, q_2, q_4

   (d) M accepts `aabb` since the final state of the computation $M(\texttt{aabb})$ is $q_4 \in F$. M accepts ε since the final state of the computation $M(\varepsilon)$ is $q_1 \in F$.

2. $M = (\{q_1, q_2, q_3, q_4\}, \{a, b\}, \delta, q_1, \{q_1, q_4\})$ where δ is defined by the following table.

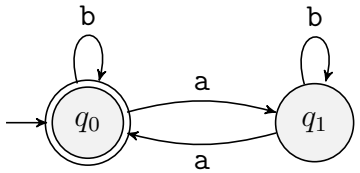
$q_i \backslash s$	a	b
q_1	q_1	q_2
q_2	q_3	q_4
q_3	q_2	q_1
q_4	q_3	q_4

3. We have the following state diagram.

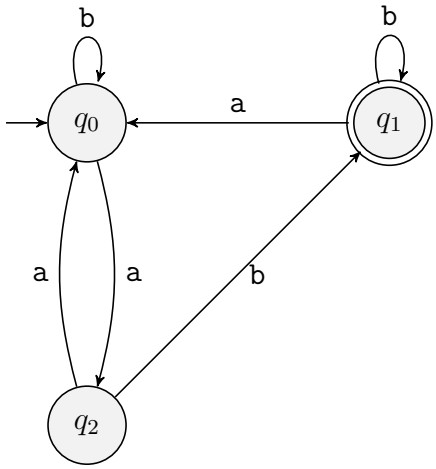


4. $q_3, q_2, q_3, q_2, q_3, q_4, q_3, q_4, q_5, q_4$ is a rejecting computation.

5. We have the following state diagram.

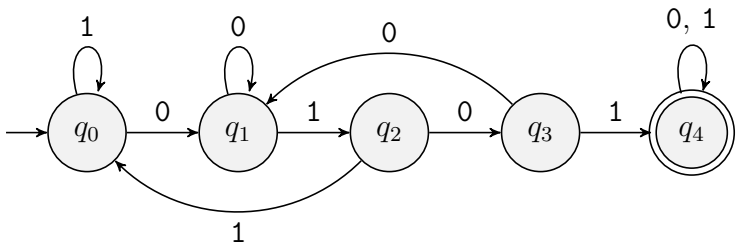


6. We have the following state diagram.

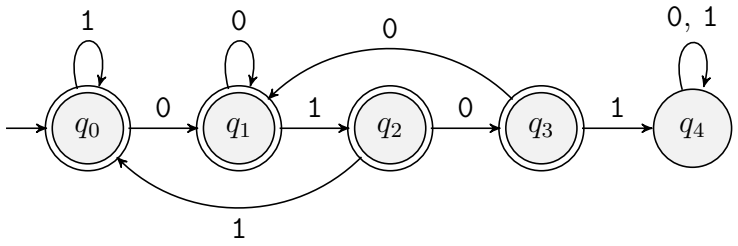


q_0: even number of a's; q_1: odd number of a's and ending with b; q_2: odd number of a's and ending with a;

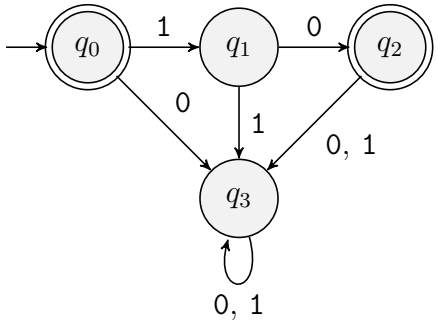
7. We have the following state diagram.



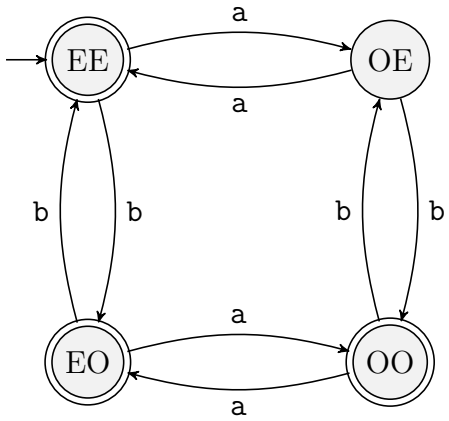
8. We have the following state diagram.



9. We have the following state diagram.

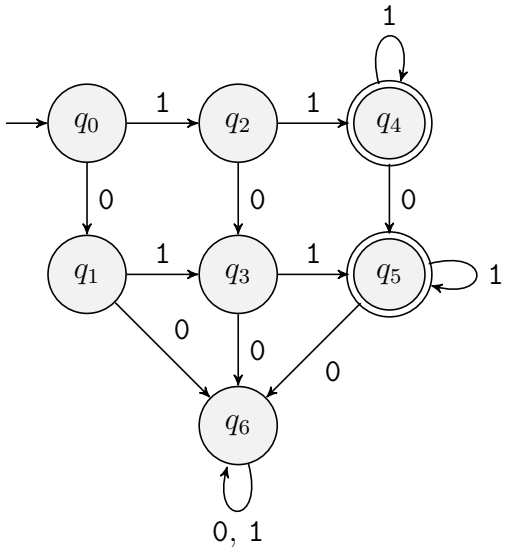


10. We have the following state diagram. For example, state EO means that an even number of a's and an odd number of b's have been read up to this point.


EE
OE
EO
OO
a
a
b
b
b
b
a
a

Additional Exercises

1. Describe the set of binary words that are accepted by the DFA given below. Prove your answer! Hint: consider the structure of the graph, in terms of the paths from initial state to accepting states.



2. Consider the alphabet

$$\Sigma = \left\{ \begin{matrix} 0 \\ 0 \end{matrix}, \begin{matrix} 0 \\ 1 \end{matrix}, \begin{matrix} 1 \\ 0 \end{matrix}, \begin{matrix} 1 \\ 1 \end{matrix} \right\}$$

where each symbol is a column vector of two bits. Provide the state diagram for a DFA that accepts all words w over Σ for which the top layer of w represents a binary number that is greater than the binary number represented by the bottom layer. For example, the DFA should accept

$$w_1 = \begin{matrix} 01011 \\ 00110 \end{matrix}$$

since the top layer represents the number 11, and the bottom layer represents the number 6. However, it should reject

$$w_2 = \begin{matrix} 10001 \\ 10011 \end{matrix}$$

since the top layer represents the number 17, the bottom layer 19, and $17 < 19$.

Solutions to Additional Exercises

1. The DFA accepts all words that have at least two 1's and at most one 0. **Proof.** Notice that accepting-state q_4 can only first be reached via the input prefix 11, while accepting-state q_5 can only first be reached either by input prefex 011 or 101. Thus, upon first reaching either accepting state with an input word, that word must have a prefix with at least two 1's and at most one 0. Moreover, any additional number of 1's keeps the computation in either q_4 or q_5. The only way to (permanently) escape both of these states is when a second 0 gets read.

2. Let $M = (\{\text{a,b,c}\}, \Sigma, \delta, \text{a}, \{\text{b}\})$, where Σ is defined by the problem statement. State a: the numbers read so far are equal, b: the top number is greater than the bottom, c: the bottom number is greater than the top. The following table provides δ.

$Q\backslash\Sigma$	$\begin{matrix}0\\0\end{matrix}$	$\begin{matrix}0\\1\end{matrix}$	$\begin{matrix}1\\0\end{matrix}$	$\begin{matrix}1\\1\end{matrix}$
a	a	c	b	a
b	b	b	b	b
c	c	c	c	c

   Once the top number has been determined to be greater (respectively, smaller) than the bottom number, it enters the permanent state b (respectively, c).