CECS 329, Exam 1 Solutions, Spring 2026, Dr. Ebert

IMPORTANT: READ THE FOLLOWING DIRECTIONS SO YOU WILL NOT LOSE POINTS. Directions: This exam has SIX different problems: one problem for each of LO's 1-3 and three additional problems.

- For each problem, write your solution using **ONE SHEET OF PAPER ONLY (BOTH FRONT AND BACK)**. Write **NAME** and **PROBLEM NUMBER** on each sheet.

- Write solutions to different problems on **SEPARATE SHEETS** of paper.

- For example, if you decide to solve all six problems, then you will submit **SIX** sheets for grading.

Unit 1 LO Problems (25 pts each)

LO1. Do the following.

(a) Consider the function `sol` which accepts two inputs: a subset of natural numbers S and a nonnegative integer t, and returns the unique subset A of S whose members sum to t. Evaluate `sol`$(\{3, 6, 8, 9, 15\}, 27)$. Also, is it true that

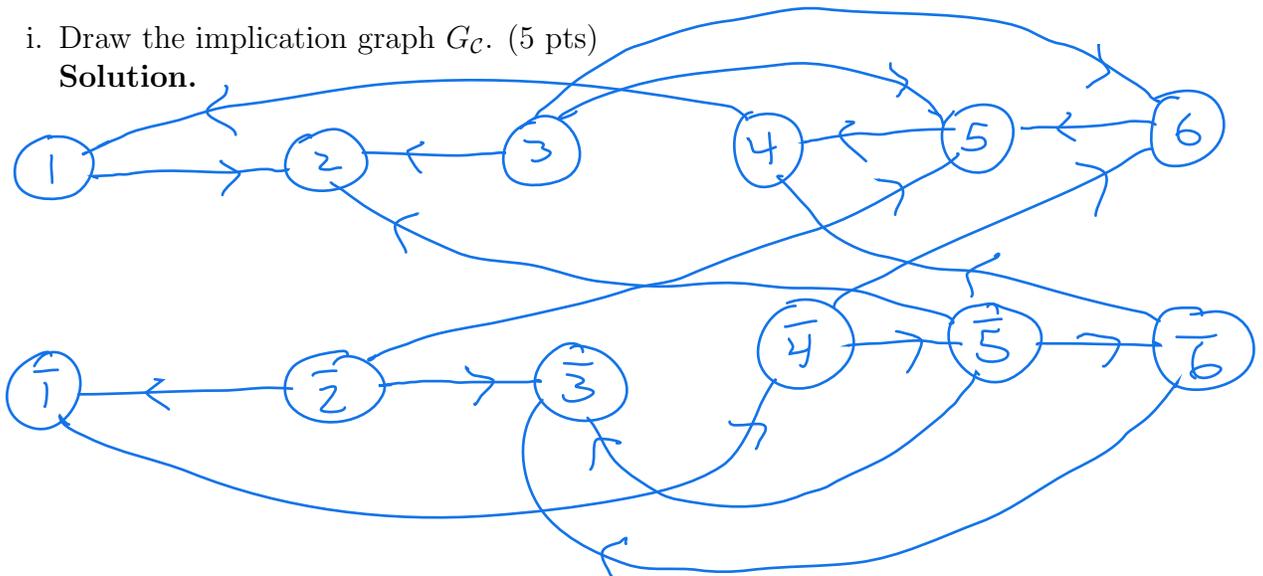$$\{\{3\}, \{2\}\} \in \{\{4\}, \{2\}, \{3\}, \{1\}\}?$$

Explain. (10 pts)

Solution. `sol`$(\{3, 6, 8, 9, 15\}, 27) = \{3, 9, 15\}$. No, $\{\{3\}, \{2\}\}$ is not a member of the set, since the set members are all singleton sets, whereas $\{\{3\}, \{2\}\}$ is a set with two members.

(b) Consider the **2SAT** instance

$$\mathcal{C} = \{(x_1, \overline{x}_4), (\overline{x}_1, x_2), (x_2, \overline{x}_3), (x_2, x_5), (\overline{x}_3, x_5), (\overline{x}_3, x_6), (x_4, \overline{x}_5), (x_4, x_6), (x_5, \overline{x}_6)\}.$$

i. Draw the implication graph $G_\mathcal{C}$. (5 pts)
   **Solution.**

ii. Perform the `Improved 2SAT` algorithm by computing the necessary reachability sets. Use numerical order (in terms of the variable index) and positive literal before negative literal when choosing the reachability set to compute next. Draw the resulting reduced 2SAT instance whenever a consistent reachability set is computed. Either provide a final satisfying assignment for $\mathcal{C}$ or indicate why $\mathcal{C}$ is unsatisfiable. (10 pts)

Solution.

$R(x_1) = \{x_1, x_2\}$ is consistent.

$\alpha_{R(x_1)} = (x_1 = 1, x_2 = 1)$

Reduced Graph:
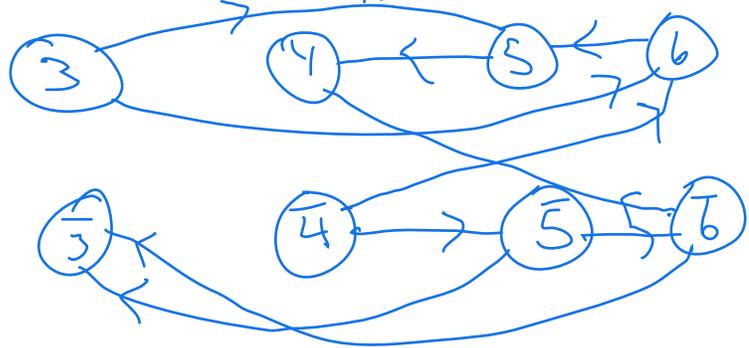
$R(x_3) = \{3, 4, 5, 6\}$ is consistent. $\sigma_{R \cdot x_3} = (x_3 = x_4 = x_5 = x_6 = 1)$

$\therefore \alpha = (x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 1, x_5 = 1, x_6 = 1)$

satisfies $\mathcal{C}$. This is easily verified, since no clause has two negative literals.

2

LO2. Do the following.

(a) Provide the general definition of what it means to be a mapping reduction from problem A to problem B. Hint: do *not* assume that A and B are decision problems. (6 pts)

Solution. See definition from mapping reducibility lecture.

(b) Recall the mapping reduction $f : \mathtt{SP} \to \mathtt{SS}$ from $\mathtt{Set\ Partition}$ to $\mathtt{Subset\ Sum}$ provided in lecture. Compute $f(S)$ for

$$S = \{2, 5, 7, 8, 9, 12, 16, 23\}.$$

(7 pts)

Solution. $f(S) = (S, t = 41)$.

(c) Verify that f is a valid mapping reduction for input S in the sense that S and $f(S)$ are either both positive instances or both negative instances of their respective decision problems. **Defend your answer**. (12 pts)

Solution. Subsets $A = \{2, 16, 23\}$ and $B = \{5, 7, 8, 9, 12\}$ partition S and both sum to 41, and thus S is a positive instance of $\mathtt{SP}$. Similarly, $A = \{2, 16, 23\}$ sums to the target $t = 41$, and thus $f(S) = (S, t = 41)$ is a positive instance of $\mathtt{SS}$.

LO3. Answer the following. Note: a minimum total of 16 points must be scored in order to pass LO3.

(a) An instance of the $\mathtt{Quadratic\ Diophantine\ (QD)}$ decision problem is a triple (a, b, c) of positive integers, and the problem is to decide there are nonnegative integers x and y for which

$$ax^2 + by = c.$$

Which of the following best describes the certificate input for a $\mathtt{QD}$ verifier that establishes $\mathtt{QD} \in \mathtt{NP}$? (6 pts)

  i. a triple of positive integers (a, b, c)
  ii. the quadratic equation $ax^2 + by = c$
  iii. a pair of nonnegative integers (x, y)
  iv. a triple of positive integers (a, b, c) *and* a pair of nonnegative integers (x, y)

Solution. Given input (a, b, c), a certificate (i.e. potential solution) takes the form of a pair (x, y), and a verifier can check its validity by evaluating $ax^2 + by = c$. Therefore, iii.

(b) An instance of the $\mathtt{Boolean\ Vector\ Sum\ (BVS)}$ decision problem is a pair (S, k), where S is a set of Boolean vectors, each having the same length, and $k \geq 1$ is a natural number. The problem is to decide if there are k different vectors $v_1, v_2, \ldots, v_k \in S$ for which

$$v_1 + v_2 + \cdots + v_k = (1\ 1\ \cdots 1),$$

where $+$ represents bitwise (inclusive) OR. For example,

$$S = \{(1\ 0\ 1\ 0\ 0), (0\ 0\ 1\ 0\ 1), (0\ 1\ 0\ 0\ 1), (0\ 1\ 0\ 1\ 1), (1\ 0\ 0\ 1\ 0)\}$$

and $k = 3$ is a positive instance of $\mathtt{BVS}$ since

$$(1\ 0\ 1\ 0\ 0) + (0\ 1\ 0\ 0\ 1) + (1\ 0\ 0\ 1\ 0) = (1\ 1\ 1\ 1\ 1).$$

3

Provide size parameters for the `BVS` decision problem. Clearly define each parameter. Hint: there are two of them. (6 pts)

Solution. $n = |S|$, while m denotes the length of each vector in S.

(c) Classify each of the following problems as being in `P`, `NP`, or `co-NP` (3 points each).

    i. An instance of decision problem `Zero Matrix Power (ZMP)` consists of an $n \times n$ matrix A having integer entries and the problem is to decide if $A^n = 0$.

    ii. An instance of decision problem `Fallacy` is a Boolean formula $F(x_1, \ldots, x_n)$, and the problem is to decide if there is no assignment α over the variables of F that can make F evaluate to 1.

    iii. An instance of decision problem `VC37` is a simple graph $G = (V, E)$ and the problem is to decide if there exists a vertex cover for G with size equal to 37.

    iv. The `Hamilton Path` decision problem presented in the Mapping Reducibility lecture.

Solution. P, co-NP, P, NP

Additional Problems (25 pts each)

A1. Let $\mathcal{C}$ be a satisfiable instance of `2SAT`, and let α be a satisfying assignment for $\mathcal{C}$. Answer the following in relation to $\mathcal{C}$.

    (a) If $P = \overline{x}_3, x_6, \overline{x}_1, \overline{x}_4, x_3$ is a path in the implication graph $G_{\mathcal{C}}$, then what can you say about satisfying assignment α? (7 pts)

    Solution. Since the truth of $\overline{x}_3$ implies the truth of x_3 (a contradiction), we must have $\alpha(x_3) = 1$.

    (b) Provide another length-4 path that must also appear in $G_{\mathcal{C}}$. Explain. (6 pts)

    Solution. Contrapositive paths also occur in the graph. $\overline{P} = \overline{x}_3, x_4, x_1, \overline{x}_6, x_3$

    (c) List all clauses that are surely in $\mathcal{C}$. Explain. (6 pts)

    Solution. Convert the implications that make the edges of P to clauses: $(x_3, x_6), (\overline{x}_6, \overline{x}_1), (x_1, \overline{x}_4), (x_4, x_3)$.

    (d) Provide two clauses that, if they were added to $\mathcal{C}$, would make $\mathcal{C}$ unsatisfiable. Explain. (6 pts)

    Solution. Add clauses that will form a path from x_3 to $\overline{x}_3$: $(\overline{x}_3, x_2)$ and $(\overline{x}_2, \overline{x}_3)$.

A2. The simple graph $G = (V, E)$ shown below and $k = 3$ together make the instance $(G, k = 3)$ of the `Clique` decision problem.
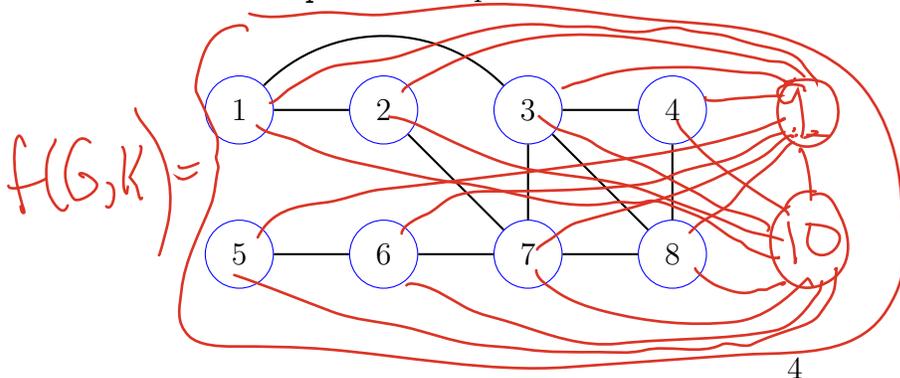
Connect $8 - (2)(3) = 2$
new vertices $(9 \& 10)$,
to every vertex $1 - 8$.
Also, connect 9 to 10.

4

(a) Provide $f(G, k)$, where f is the mapping reduction from `Clique` to `Half Clique` provided in the core exercises of the Mapping Reducibility lecture. (10 pts) Note: correctly solving this problem counts for passing one half of LO2.

Solution. See above.

(b) Show that f is a valid mapping reduction for input (G, k) in the sense that (G, k) and $f(G, k)$ are either both positive instances or both negative instances of their respective decision problems. **Defend your answer**. (15 pts)

Solution. $(G, k = 3)$ is a positive instance since $\{3, 4, 8\}$ is a 3-clique for G. Also, $\{3, 4, 8, 9, 10\}$ is a 5-clique for $f(G, k)$ and hence $f(G, k)$ has a half clique since five is one half of ten (the number of vertices for $f(G, k)$). So, $f(G, k)$ is a positive instance and the mapping is valid for $(G, k = 3)$.

A3. Recall the `Vertex Cover (VC)` decision problem that was defined in the Mapping Reducibility lecture. Let $G = (V, E)$ and $k \geq 0$ be an instance of `VC`. Let C be a subset of k vertices that serves as a certificate for (G, k).

(a) Provide pseudocode for a verifier program that takes as inputs both (G, k) and C, and returns 1 iff C is a vertex cover of size k for G. (15 pts)

Solution.

> For each $e = (u, v) \in E$,
>> found $= 0$.
>> For each $w \in C$,
>>> If $(w == u \lor w == v)$, then found $= 1$ and `break`.
>> If found $== 0$, then return 0.
>
> Return 1.

(b) Provide size parameters for `VC` and use them to describe the number of steps required by your verifier from part a. Defend you answer. (10 pts)

Solution. $n = |V|$, $m = |E|$. The outer `for`-loop iterates $O(m)$ times. For each outer iteration, the inner `for`-loop iterates $O(n)$ times. Moreover, each inner loop requires $O(1)$ steps. Putting these facts together, we see that the worst-case running time is $O(mn)$.