CECS 528, Quiz 3, Spring 2026, Dr. Ebert

IMPORTANT: READ THE FOLLOWING DIRECTIONS.

- For each LO, write your solutions to all parts using **ONE SHEET OF PAPER ONLY (BOTH FRONT AND BACK)**. Write **NAME** and **PROBLEM NUMBER** on each sheet.
- Use **SEPARATE SHEETS** for different LO/Additional problems.
- You may solve an amount of problems that sums to **AT MOST THREE WHOLE PROBLEMS**. Two parts of different LO's count for one whole problem. For example, solving LO1 part b, LO2 part a, LO3 part a, LO4 part a, and LO5 counts as three whole problems. Further solutions that go beyond this amount will be ignored.

LO Makeup Problems

LO1. Solve the following problems.

(b) Use the substitution method to prove that, if $T(n)$ satisfies

$$T(n) = 2T(n/4) + 5\sqrt{n},$$

then $T(n) = O(\sqrt{n}\log n)$.

LO2. Solve the following problems.

(a) For the `Find Statistic` algorithm, describe **in just a few sentences** how the pivot M is obtained for the partitioning step of the algorithm. Also, the inequality

$$3(\lfloor\frac{1}{2}\lceil\frac{1}{5}n\rceil\rfloor - 2) \geq 3(\frac{1}{2}\cdot\frac{n}{5} - 3) = \frac{3n}{10} - 9$$

was provided to establish that, for the partitioning step, about 30% of the array elements are less than or equal (respectively, greater than or equal) to M. Explain the significance of each of the following factors that make up the left expression: i) $\frac{n}{5}$ ii) $\frac{1}{2}$, iii) 3. Provide a few sentences for each one.

(b) Demonstrate the partitioning step of Hoare's Quicksort for the array

$$a = 16, 2, 1, 5, 6, 3, 13, 10, 7, 12, 18, 10, 19.$$

Use the median-of-three method to select the pivot.

LO3. Do the following.

(a) The divide-and-conquer FFT algorithm provides an efficient way to simultaneously evaluate a degree-$(n-1)$ polynomial at n different inputs, where n is a power of two. i) What do we call these n inputs? ii) Provide the equation that relates $A(x)$ to the two subproblem polynomials $A_e(x)$ and $A_o(x)$. iii) Based on your answers to i) and ii), what special property must each of these n inputs possess?

What are the degrees of these two polynomials? Based on this equation, why is it essential that, for even n, the nth roots of unity come in additive-inverse pairs?

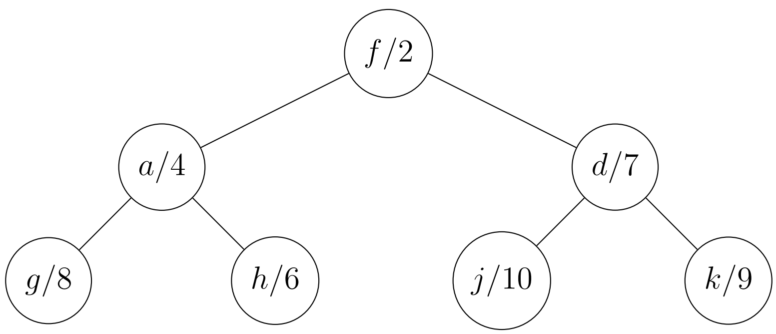
(b) If $p(x) = 5 + 3x - 4x^2 - 2x^3$, then compute $\text{DFT}^{-1}(p)$ using the IFFT algorithm. Show the entire recursion tree as was done in the lecture notes.

LO4. Do the following.

(a) The tree below shows the state of the binary min-heap at the beginning of some round of Dijkstra's algorithm, applied to some weighted *directed* graph G. If G has edges

$$(f, d, 1), (a, f, 6), (d, g, 4), (f, g, 3), (f, h, 2), (f, k, 3),$$

then draw a plausible state of the heap at the end of the round.



(b) State the greedy choice that is being made in each round of Prim's algorithm. The weighted edges of a graph $G = (V, E)$ are

$$E = \{(1, 2, 11), (1, 3, 19), (1, 4, 15), (1, 5, 13), (1, 6, 8), (2, 3, 14), (2, 4, 17),$$

$$(2, 5, 16), (2, 6, 22), (3, 4, 10), (3, 5, 12), (3, 6, 19), (4, 5, 15), (5, 6, 20)\}.$$

For each round of Prim's algorithm applied to G, indicate the selection for that round and provide a drawing of the final output of the algorithm. Hint: you do *not* need to use a heap data structure. Also, break any ties by choosing the vertex having least value. For example, if there is a tie beetween vertices 2 and 4, then choose vertex 2.

LO5. Do the following.

(a) The dynamic-programming algorithm that solves the `Optimal Binary Search Tree (OBST)` optimization problem defines a recurrence for the function $\text{wac}(i, j)$. State in words the meaning of $\text{wac}(i, j)$ and provide its recurrence.

(b) Use the recurrence from part a) to solve the instance of `OBST` that has keys 1,2,3,4 with respective weights 10,20,40,50. Provide the matrix of subproblem solutions (including the value of k associated with each subproblem instance (of size at least two). Draw the optimal binary search tree.

LO6. Given the 2SAT instance

$$\mathcal{C} = \{(x_1, \overline{x}_3), (\overline{x}_1, x_2), (\overline{x}_1, x_3), (\overline{x}_1, x_4), (x_2, x_3), (x_2, \overline{x}_6), (\overline{x}_2, \overline{x}_4), (\overline{x}_3, x_4), (x_4, \overline{x}_5), (x_5, x_6)\},$$

do the following.

(a) Draw the implication graph $G_{\mathcal{C}}$.

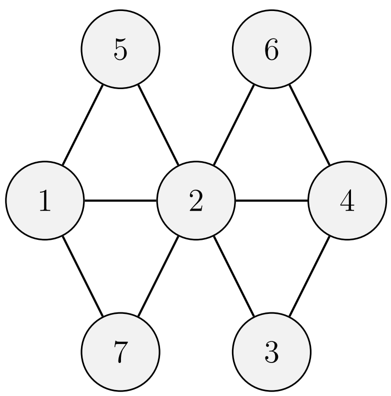
(b) Perform the `Improved 2SAT` algorithm by computing the necessary reachability sets. Use numerical order (in terms of the variable index) and positive literal before negative literal when choosing the reachability set to compute next. Draw the resulting reduced 2SAT instance whenever a consistent reachability set is computed. Either provide a final satisfying assignment for $\mathcal{C}$ or indicate why $\mathcal{C}$ is unsatisfiable.

(c) Given 2SAT instance $\mathcal{C}$, suppose the reachability set $R(x_2)$ contains both x_1 and $\overline{x}_1$. Explain why $R(x_2)$ must also contain $\overline{x}_2$. Hint: contrapositive.

LO7. Answer the following.

(a) Provide the definition of what it means to be a mapping reduction from decision problem A to decision problem B.

(b) For the mapping reduction f : `Vertex Cover` $\rightarrow$ `Half Vertex Cover`, draw $f(G, k)$ for the `Vertex Cover` instance whose graph is shown below, and for which $k = 3$.



(c) Verify that both (G, k) and $f(G, k)$ are either both positive instances, or are both negative ones. Explain and show work.

LO8. Do the following.

(a) An instance of `Set Packing` is a positve integer $k > 0$ and a collection of sets $\mathcal{S}$, where each set $C \in S$ is a subset of $\{1, \ldots, n\}$ for some $n \geq 1$. The problem is to decide if there are k sets $C_1, \ldots, C_k$ in $\mathcal{S}$ that are **pairwise disjoint** meaning that, for every $1 \leq i < j \leq n$, $C_i \cap C_j = \emptyset$. Thus a natural certificate for `Set Packing` is an array a of length k, where $a[i] \in \mathcal{S}$, for $i = 1, \ldots, k$. To show that `Set Packing` is an NP problem, do the following.

i. Provide the pseudocode for a verifier program for `Set Packing` that takes as inputs $\mathcal{S}$, k, and a and returns 1 iff the members of a are pairwise disjoint.

ii. Given that the size parameters for `Set Packing` are $m = |\mathcal{S}|$ and n, determine the big-O number steps required by your verifier. Defend your answer. Hint: your analysis should accurately address the number of steps required to check the intersection of two sets.

(b) Classify each of the following problems as being in `P`, `NP`, or co-`NP`. Note: at least three correct answers is necessary for passing this part of LO8.

i. An instance of `Quadratic Congruences` is a triple of positive integers (a, b, c) and the problem is to decide if there is a positive integer $x < c$ for which $x^2 \bmod b = a$. Hint: the size of the problem instance is $\lfloor \log(\max(a, b, c)) \rfloor + 1$.

ii. An instance of `Increasing Subsequence` is an array a of size n and a positive integer k the problem is to decide if there are indices $0 \leq i_1 < i_2 < \cdots < i_k < n$ for which $a[i_1] < a[i_2] < \cdots < a[i_k]$.

iii. An instance of `Longest Common Subsequence` is a pair of words (w_1, w_2) each over some alphabet Σ, and a nonnegative integer $k \geq 0$. The problem is to decide if there is a word v over Σ that has length k and appears in both w_1 and w_2, where the appearance can be non-contiguous. For example, the word `coin` appears non-contiguously in the word **co**nctentat**i**o**n**.

iv. An instance of the `Strongly Cyclic` decision problem is a pair (G, k) where $G = (V, E)$ is a directed graph and k is a nonnegative integer. The problem is to decide if G always has at least one cycle after k vertices (and the edges that are incident with them) are removed from G, regardless of what k vertices are selected.