Fast Fourier Transform

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1 Introduction

Like Strassen's algorithm, the Fast Fourier Transform (FFT) is considered one of the more suprising and interesting known divide-and-conquer algorithms. It finds important use in the field of signal and image processing but is perhaps best understood as a means for efficiently multiplying two polynomials which we present in this lecture.

2 Polynomial Multiplication and the Fast Fourier Transform

Given two polynomials

$$A(x) = a_0 + a_1 x + \cdots + a_d x^d$$

and

$$B(x) = b_0 + b_1 x + \cdots + b_d x^d,$$

our goal is to compute the product $C(x) = A(x)B(x)$ where $C(x)$ is a degree-$2d$ polynomial whose k th term c_k, $k = 0, 1, \ldots, 2d$, is computed as

$$c_k = \sum_{i=0}^{k} a_i b_{k-i}.$$

Thus, using the above formula we see that computing the first $d + 1$ coefficients of $C(x)$ requires

$$1 + 2 + 3 + 4 + \cdots + d + (d + 1) = \Theta(d^2)$$

steps.

The following algorithm provides an alternative way to compute $C(x)$.

Alternative Polynomial Multiplication Algorithm

Input: Coefficients of polynomials $A(x)$ and $B(x)$.

Output: Coefficients of $C(x) = A(x)B(x)$.

Pick points: $x_0, x_1, \ldots, x_{n-1}$, for some $n \geq 2d + 1$.

Evaluate A and B: compute $A(x_0), \ldots, A(x_{n-1})$ and $B(x_0), \ldots, B(x_{n-1})$.

Evaluate C: compute $C(x_0) = A(x_0)B(x_0), \ldots, C(x_{n-1}) = A(x_{n-1})B(x_{n-1})$.

Interpolate: determine the unique coefficients $c_0, c_1, \ldots, c_{2d}$ for which, for all $i = 0, 1, \ldots, n-1$,

$$C(x_i) = c_0 + c_1 x_i + \cdots + c_{2d} x_i^{2d}.$$

Return $c_0, c_1, \ldots, c_{2d}$.

Notes:

1. On the surface, it appears that this method will also require $O(d^2)$ steps, since evaluating a d-degree polynomial on some input x_i generally requires $\Theta(d)$ steps via Horner's algorithm.
2. Moreover, interpolation also requires $O(d^2)$ steps since, as we'll see, it involves inverting a $2d \times 2d$ Vandermonde matrix.
3. However, by choosing to *simultaneously* evaluate A (and B) with the n inputs via a divide-and-conquer approach, the total number of evaluation and interpolation steps may be reduced to $O(n \log n)$.

2.1 A Divide and Conquer approach to polynomial evaluation

In what follows we assume that n is a power of two. Consider the polynomial

$$A(x) = \underline{a_0} + a_1x + \underline{a_2}x^2 + \cdots + a_{n-1}x^{n-1}.$$

$$\begin{tikzcd} 1 \arrow[d] & x^2 \arrow[d] & x^4 \arrow[d] & x^6 \arrow[d] & \cdots \\ 1 & y & y^2 & y^3 & \end{tikzcd}$$

Then $A(x)$ may be written as

$$A(x) = A_e(x^2) + xA_o(x^2),$$

where $A_e(y)$ and $A_o(y)$ are the polynomials

$$A_e(y) = \underline{a_0} + \underline{a_2}y + \underline{a_4}y^2 + \cdots + \underline{a_{n-2}}y^{\frac{n-2}{2}},$$

$$\begin{tikzcd} & x & x^3 & x^5 & \cdots \\ x(& 1 \arrow[d] & x^2 \arrow[d] & x^4 \arrow[d] &) \\ & 1 & y & y^2 & \end{tikzcd}$$

and

$$A_o(y) = a_1 + a_3y + \cdots + a_{n-1}y^{\frac{n-2}{2}}.$$

Thus, we may evaluate $(n-1)$-degree polynomial $A(x)$ by evaluating two $(\frac{n-2}{2})$-degree polynomials at x^2. In other words, we've taken the problem and divided it into two subproblems, each of which is one-half the size. This yields the recurrence

$$\underline{T(n) = 2T(n/2) + n,}$$

where $f(n) = n$ accounts for the fact that there are n inputs to $A(x)$, each of which may be computed in O(1) steps using the equation

$$\boxed{A(x) = A_e(x^2) + xA_o(x^2).}$$

Therefore, assuming an appropriate choice for the n inputs to $A(x)$, the algorithm requires $O(n \log n)$ steps.

Example 2.1. For

$$A(x) = \underline{-2} + \underline{5x} + \underline{3x^2} - \underline{4x^3} + \underline{9x^4} + \underline{x^5} - \underline{8x^6} - \underline{2x^7},$$

compute both $A_e(y)$ and $A_o(y)$.

$$A_e(y) = -2 + 3y + 9y^2 - 8y^3$$

$$A_o(y) = 5 - 4y + y^2 - 2y^3$$

$$A_e(x^2) = -2 + 3x^2 + 9x^4 - 8x^6$$

Upon examining the formula $-3\ 3\ -2\ 2\ -1\ 1$

$$A(x) = A_e(x^2) + xA_o(x^2),$$

in order for the divide-and-conquer strategy to work, the following to properties must be true about the n numbers

$$x_1, x_2, \ldots, x_{\frac{n}{2}}, x_{\frac{n}{2}+1}, x_{\frac{n}{2}+2}, \ldots, x_n$$

that we use to evaluate $A(x)$.

1. Without loss of generality, we should have $x_{\frac{n}{2}+i} = -x_i$, i.e. if x_i is in the list, then so is its additive inverse. Why? Because the answers to $A_e(x^2)$ and $A_o(x^2)$ may be used by both x_i and $-x_i$, since $x_i^2 = (-x_i)^2$. Thus we only need to evaluate A_e and A_o a number of times that is one-half the number of inputs. In other words, if the problem is to evaluate a degree-$(n-1)$ polynomial at n inputs, then a problem instance that has a polynomial of degree $(n-2)/2$ should require the evaluation of $\frac{n}{2}$ inputs.

2. The first property should recursively also hold for $x_1^2, x_2^2, \ldots, x_{\frac{n}{2}}^2$. Why? Because these numbers will serve as the inputs to the $n/2$-sized subproblems A_e and A_o which are the two subproblems of our divide-and-conquer algorithm. In general, the first property must hold for *any* subproblem of any size that occurs as one of the problem instances of the divide-and-conquer algorithm.

The problem is that, in the case of a real number, its square is nonnegative, and so Property 1 will not hold when moving down to level 1 of the recursion (here we assusme that level 0 is the root level). However, Property 1 *does* hold for the nth roots of unity which are complex numbers that we now define below.

3 Roots of Unity

Euler's Identity. For any real number θ, $e^{i\theta} = \cos\theta + i\sin\theta$

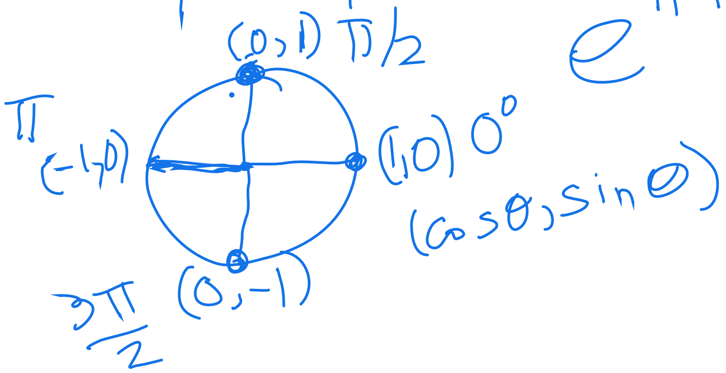
Example 3.1. Evaluate $e^{i\theta}$ for selected values of θ.

$$e^{i\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = 0 + i\cdot 1 = \boxed{i}$$

$i^2 = -1$ $\quad i^3 = -i$

$i^4 = 1$

$$e^{\pi i} = \cos(\pi) + i\sin(\pi) = \boxed{-1}$$



Complex *n*th roots of unity. For each $j = 0, \dots, n-1$, $e^{\frac{2\pi ij}{n}}$ is a **complex *n*th root of unity**, meaning that

$$e^{\frac{2\pi i j}{n}}$$

$$e^{(\frac{2\pi ij}{n})^n} = e^{2\pi ij} = \cos(2\pi j) + i\sin(2\pi j) = 1.$$

Example 3.2. Determine the complex 4th roots of unity.

Solution.

$$e^{\frac{2\pi i 0}{4}} = \boxed{1} \qquad e^{\frac{2\pi i\cdot 1}{4}} = e^{\frac{\pi}{2}i} = \boxed{i}$$

$$e^{\frac{2\pi i\cdot 2}{4}} = e^{\pi i} = \boxed{-1} \qquad e^{\frac{2\pi i\cdot 3}{4}} = e^{\frac{3\pi}{2}i} = \boxed{-i}$$

$1, i, -1, -i$

$n=2$ roots of unity $1, -1$

$1^2 \quad i^2 \quad -1^2 \quad -i^2 = 1, -1, 1, -1$

The next proposition shows that $e^{\frac{2\pi ij}{n}}$, $\underline{j = 0, \ldots, n-1}$, are the only unique powers of $e^{\frac{2\pi i}{n}}$.

Proposition 3.3. If integers j and k satisfy $j \equiv k \bmod n$, then

$$e^{\frac{2\pi ij}{n}} = e^{\frac{2\pi ik}{n}}.$$

$$e^{\frac{2\pi i n}{n}} = 1$$

Proof of Proposition. Assume $j \equiv k \bmod n$. Then $k = nq + j$, for some integer q. Then

$$e^{\frac{2\pi ik}{n}} = e^{\frac{2\pi i(j+nq)}{n}} = e^{\frac{2\pi ij}{n}} e^{\frac{2\pi inq}{n}} = e^{\frac{2\pi ij}{n}} e^{2\pi iq} = e^{\frac{2\pi ij}{n}} \cdot 1 = e^{\frac{2\pi ij}{n}}.$$

Definition 3.4. For $j = 0, \ldots, n-1$, ω_n^j denotes the j th root of unity $e^{\frac{2\pi ij}{n}}$.

$$e^{\frac{2\pi ij}{n}} = \omega_n^j$$

Proposition 3.5. The n th roots of unity form an **abelian group** under multiplication. In other words, the following properties hold for all integers i, j and k. Note: all exponents and exponent arithmetic is assumed as mod n arithmetic.

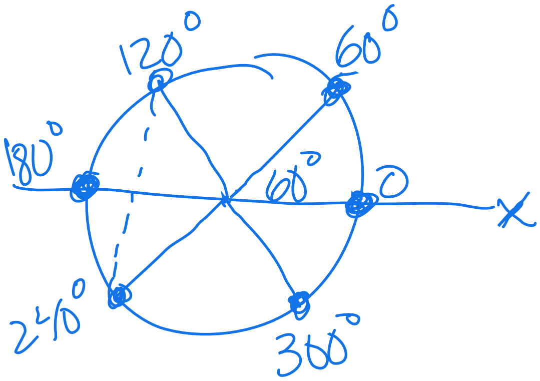
Associativity $(\omega_n^i \cdot \omega_n^j) \cdot \omega_n^k = \omega_n^i \cdot (\omega_n^j \cdot \omega_n^k) = \omega_n^{i+j+k}$.

Commutative $\omega_n^i \cdot \omega_n^j = \omega_n^j \cdot \omega_n^i = \omega_n^{i+j}$.

Existence of Unit $\omega_n^0 = 1$ and $1 \cdot \omega_n^i = \omega_n^i$.

Existence of Multiplicative Inverse $(\omega_n^i)^{-1} = \omega_n^{-i}$, since $\omega_n^i \cdot \omega_n^{-i} = \omega_n^{i-i} = \omega_n^0 = 1$.

Example 3.6. For the 6th roots of unity, determine the multiplicative inverse of $\frac{\sqrt{3}}{2} + \frac{1}{2}i$. Verify your answer by writing the inverse in standard form and multiplying with $\frac{1}{2} + \frac{\sqrt{3}}{2}i$.



$$1, \frac{1}{2} + \frac{\sqrt{3}}{2} i, \underline{-\frac{1}{2} + \frac{\sqrt{3}}{2}}, -1, -\frac{1}{2} - \frac{\sqrt{3}}{2} i, \frac{1}{2} - \frac{\sqrt{3}}{2} i$$

$$\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)\left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = \frac{1}{4} + \frac{3}{4} = 1$$

Proposition 3.7. When $n \geq 2$ is even then the nth roots of unity have the following properties.

1. ω_n^j and $-\omega_n^j$ are both roots of unity. In other words, roots of unity come in additive-inverse pairs. Furthermore, if $0 \leq j < n/2$, then $\omega_n^{j+n/2} = -\omega_n^j$.
2. the squares of the nth roots of unity yield the $n/2$ roots of unity.

Proof of Proposition.

1. By the sum-of-angle formulas for cosine and sine, we have

   $$e^{(\theta+\pi)i} = \cos(\theta+\pi) + i\sin(\theta+\pi) = -\cos\theta - \sin\theta i = -e^{\theta i}.$$

   Therefore,

   $$-\omega_n^j = e^{(\frac{2\pi ij}{n}+\pi i)} = e^{(\frac{2\pi ij}{n}+\frac{2\pi i(n/2)}{n})} = e^{\frac{2\pi i(j+n/2)}{n}} = \omega_n^{j+(n/2)}$$

   which is a root of unity.

2. For $0 \leq j < n/2$, we have

   $$(\omega_n^j)^2 = \omega_n^{2j} = e^{\frac{2\pi i(2j)}{n}} = e^{\frac{2\pi ij}{n/2}},$$

   which is an $n/2$ root of unity. Note also that, for $n/2 \leq j < n$, $e^{\frac{2\pi ij}{n}}$ is just the negative of ω_n^j, and thus its square yields the same $n/2$ root of unity as its additive-inverse counterpart.

□

The above divide-and-conquer algorithm leads us to the following definition.

Definition 3.8. Given complex coefficients $c_0, \ldots, c_{n-1}$, let $p(x)$ be the polynomial

$$p(x) = \sum_{k=0}^{n-1} c_k x^k.$$

Then the n**th order discrete Fourier transform** is the function

$$\mathrm{DFT}_n\underline{(c_0, \ldots, c_{n-1})} = \underline{(y_0, \ldots, y_{n-1})},$$

where $y_j = p(\omega_n^j)$, $j = 0, \ldots, n-1$.

In words the nth order discrete Fourier transform, takes as input the complex coefficients of a degree $n-1$ polynomial p, and returns the n-dimensional vector whose components are the evaluation of p at each of the nth roots of unity. Another way to write $\mathrm{DFT}_n(c_0, \ldots, c_{n-1})$ is $\mathrm{DFT}_n(p)$, where p is the polynomial of degree $n-1$ whose coefficients are $c_0, \ldots, c_{n-1}$.

Example 3.9. Compute $\mathrm{DFT}_4(0, 1, 2, 3)$.

$$A(x) = 0 + x + 2x^2 + 3x^3$$

$n=4$ roots of unity

$1, i, -1, -i$

$$A(1) = \boxed{6}$$

$$A(i) = 0 + i + 2i^2 + 3i^3 = \quad (-i)^3$$

$$\boxed{-2-2i}$$

$$A(-1) = \boxed{-2} \qquad A(-i) = -i - 2 + 3i =$$

$$\boxed{-2+2i}$$

$$\mathrm{DFT}_4(A(x)) = (6, -2-2i, -2, -2+2i)$$

3.1 Fast Fourier Transform

We may now write our divide-and-conquer algorithm in terms of DFT_n. In what follows we define

$$(u_1, \ldots, u_n) \odot (v_1, \ldots, v_n) = (u_1 v_1, \ldots, u_n v_n),$$

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which we call the **scaling of v with u**.

Fast Fourier Transform

```
Input: polynomial $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$, where $n$ is a power of two.
Output: $\mathrm{DFT}_n(A)$.
If $n = 1$, then return $(a_0)$.
$Y_0 = \mathrm{DFT}_{\frac{n}{2}}(A_e)$.
$Y_0 = Y_0 \circ Y_0$. //Concatenate vector $Y_0$ with itself.
$Y_1 = \mathrm{DFT}_{\frac{n}{2}}(A_o)$.
$Y_1 = Y_1 \circ Y_1$. //Concatenate vector $Y_1$ with itself.
$Y_1 = \vec{\omega_n} \odot Y_1$. //Scale $Y_1$ with the length-$n$ vector of $n$th roots of unity.
Return $Y_0 + Y_1$. //Return the vector sum of $Y_0$ with $Y_1$.
```

$\vec{\omega}_n$ = nth roots of unity as a vector

Y_1

||

$\hookrightarrow \boxed{A(x) = A_e(x^2) + x A_o(x^2)}$

We see that the running time for FFT is $\Theta(n \log n)$, since its running time satisfies

$$T(n) = 2T(n/2) + n.$$

Thus, we have found a way to evaluate a polynomial at n points using only a log-linear number of steps!

Example 3.10. Compute $\mathrm{DFT}_4(0,1,2,3)$ using the FFT algorithm.

Solution. $\mathrm{DFT}_4(0,1,2,3) =$

$(2,-2,2,-2) + (1,i,-1,-i) \odot (4,-2,4,-2) = \boxed{(6,-2-2i,-2,-2+2i)}$

$\mathrm{DFT}_2(0,2) =$

$(0,0) + (1,-1) \odot (2,2) = (2,-2)$

$\mathrm{DFT}_2(1,3) =$

$(1,1) + (1,-1) \odot (3,3) = (4,-2)$

$\mathrm{DFT}_1(0) = 0 \quad \mathrm{DFT}_1(2) = 2$

$\mathrm{DFT}_1(1) = 1 \quad \mathrm{DFT}_1(3) = 3$

4 Solving Interpolation with the Inverse DFT

Returning to the alternative polynomial multiplication algorithm, the FFT algorithm allows us to compute $C(\omega_n^j)$, for each $j = 0, 1, \ldots, n-1$. To finish the algorithm, we must find coefficients $c_0, c_1, \ldots, c_{n-1}$ for which, for each $j = 0, 1, \ldots, n-1$,

$C(1) = A(1)B(1)$
$C(i) = A(i)B(i)$

$$C(\omega_n^j) = c_0 + c_1\omega_n^j + \cdots + c_{n-1}\omega_n^{j(n-1)}.$$

Furthermore, we can write these n equations in matrix form as follows.

$n=4$

$C(1) = C_0 + C_1 + C_2 + C_3$
$C(i) = C_0 + iC_1 + (i)^2C_2 + (i)^3C_3$

Coefficients of C

Solve for these

$$\begin{pmatrix} C(\omega_n^0) \\ C(\omega_n^1) \\ \vdots \\ C(\omega_n^{n-1}) \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega_n^{1\cdot 1} & \cdots & \omega_n^{1(n-1)} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \omega_n^{n-1} & \cdots & \omega_n^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{pmatrix}$$

Letting F_n denote the $n \times n$ matrix in the above equation, we leave it as an exercise to show that its inverse is

$$F_n^{-1} = \frac{1}{n} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega_n^{-1\cdot 1} & \cdots & \omega_n^{-1(n-1)} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \omega_n^{-(n-1)} & \cdots & \omega_n^{-(n-1)(n-1)} \end{pmatrix}.$$

$\vec{C} = F_n\vec{c}$
$F_n^{-1}\vec{C} = F^{-1}F\vec{c}$
$\boxed{F_n^{-1}C} = I\vec{c} = \vec{c}$

Thus, we may compute the coefficients of $C(x)$ as

$$\underline{\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{pmatrix}} = \frac{1}{n} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega_n^{-1} & \cdots & \omega_n^{-1(n-1)} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \omega_n^{-(n-1)} & \cdots & \omega_n^{-(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} C(\omega_n^0) \\ C(\omega_n^1) \\ \vdots \\ C(\omega_n^{n-1}) \end{pmatrix}.$$

Thus, for all $j = 0, 1, \ldots, n-1$, we have

$$c_j = \frac{1}{n}(C(\omega_n^0) + C(\omega_n^1)\omega_n^{-j} + \cdots + C(\omega_n^{n-1})\omega_n^{-j(n-1)}).$$

Notice that this equation is essentially the evaluation of polynomial

$$\frac{1}{n}(C(\omega_n^0) + C(\omega_n^1)x + \cdots + C(\omega_n^{n-1})x^{n-1})$$

on input $x = \omega_n^{-j}$. This suggests the following definition.

Definition 4.1. Given complex coefficients $y_0, \ldots, y_{n-1}$, let $p(x)$ be the polynomial

$$p(x) = \sum_{k=0}^{n-1} y_k x^k.$$

Then the n**th order inverse discrete Fourier transform** is the function

$$\mathrm{DFT}_n^{-1}(y_0, \ldots, y_{n-1}) = (c_0, \ldots, c_{n-1}),$$

where $\boxed{c_j = \underline{\tfrac{1}{n}}p(\omega_n^{-j}),}$ $j = 0, \ldots, n-1$.

In words the nth order inverse discrete Fourier transform, takes as input the complex coefficients of a degree $n-1$ polynomial p, and returns the n-dimensional vector whose components are the evaluation of $\frac{1}{n}p(x)$ at each of the inverses of the nth roots of unity.

4.1 The Inverse Fast Fourier Transform

We may provide a similar divide-and-conquer algorithm for computing DFT_n^{-1} which we call the **Inverse Fast Fourier Transform (IFFT)**.

Inverse Fast Fourier Transform

```
Input: polynomial $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$, where $n$ is a power of two.
Output: $\text{DFT}_n^{-1}(A)$.
If $n = 1$, then return $(a_0)$.
$Y_0 = \text{DFT}_{\frac{n}{2}}^{-1}(A_e)$.
$Y_0 = Y_0 \circ Y_0$. //Concatenate vector $Y_0$ with itself.
$Y_1 = \text{DFT}_{\frac{n}{2}}^{-1}(A_o)$.
$Y_1 = Y_1 \circ Y_1$. //Concatenate vector $Y_1$ with itself.
$Y_1 = \vec{\omega}_n^{-1} \odot Y_1$. //Scale $Y_1$ with the respective inverses of the $n$th roots of unity.
Return $\frac{1}{2}(Y_0 + Y_1)$. //Return the vector sum of $Y_0$ with $Y_1$.
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$$\frac{1}{2^{\lg n}} = \boxed{\frac{1}{n}}$$

Notice that in the final line we must scale the vector by 1/2. This is because both $\text{DFT}_{\frac{n}{2}}^{-1}(A_e)$ and $\text{DFT}_{\frac{n}{2}}^{-1}(A_o)$ give the polynomial evaluations divided by $n/2$. However, we want both to be divided by n. So we must multiply by $n/2$ to undo the division by $n/2$, and then divide by n, which has the net effect of multiplying by $1/2$.

$A(x) = x - x^2 + 2x^3 \quad A(1) = 2 \quad A(-i) = -i + 1 + 2i = 1 + i$

$A(-1) = -4 \quad A(i) = 1 - i$

Example 4.2. Compute $\mathrm{DFT}_4^{-1}(0, 1, -1, 2)$ by a) using the definition of $\mathrm{DFT}_4^{-1}(0, 1, -1, 2)$, and b) using the IFFT algorithm on $\mathrm{DFT}_4^{-1}(0, 1, -1, 2)$.

$$\mathrm{DFT}_4^{-1}(A) = \frac{1}{4}\left(2, 1+i, -4, 1-i\right) = \left(\frac{1}{2}, \frac{1}{4} + \frac{i}{4}, -1, \frac{1}{4} - \frac{i}{4}\right)$$

$$\mathrm{DFT}_4^{-1}(0, 1, -1, 2) = \frac{1}{2}\left(\left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) + (1, -i, -1, i) \odot \left(\frac{3}{2}, -\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}\right)\right) = \left(\frac{1}{2}, \frac{1}{4} + \frac{1}{4}i, -1, \frac{1}{4} - \frac{1}{4}i\right)$$

$$\mathrm{DFT}_2^{-1}(0, -1) = \frac{1}{2}\left((0, 0) + (1, -1) \odot (-1, -1)\right) = \left(-\frac{1}{2}, \frac{1}{2}\right)$$

$$\mathrm{DFT}_2^{-1}(1, 2) = \frac{1}{2}\left((1, 1) + (1, -1) \odot (2, 2)\right) = \left(\frac{3}{2}, -\frac{1}{2}\right)$$

$\mathrm{DFT}_1^{-1}(0) = 0 \quad \mathrm{DFT}_1^{-1}(1) = 1 \qquad \mathrm{DFT}_1^{-1}(1) = 1 \quad \mathrm{DFT}_1^{-1}(2) = 2$

4.2 Summary

$DFT_n(p)$ The *discrete Fourier transform* that evaluates an $(n-1)$-degree polynomial p at each of the nth roots of unity and returns a vector of these evaluations.

FFT An *algorithm* for computing $DFT_n(p)$ in $\mathrm{O}(n \log n)$ steps when n is assumed a power of 2.

$DFT_n^{-1}(p)$ The *inverse discrete Fourier transform* that evaluates an $(n-1)$-degree polynomial p at each multiplicative inverse of each nth root of unity, and returns a vector of these evaluations scaled by $\frac{1}{n}$. Moreover if the coefficients of p are the values $q(\omega_n^0), q(\omega_n^1), \ldots, q(\omega_n^{n-1})$, for some $(n-1)$-degree polynomial q, then $DFT_n^{-1}(p)$ outputs the coefficients of q, meaning that it solves the problem of *polynomial interpolation* with respect to q

IFFT An *algorithm* for computing $DFT_n^{-1}(p)$ in $\mathrm{O}(n \log n)$ steps when n is assumed a power of 2.

FFT Core Exercises

1. Compute $\mathrm{DFT}_4(1, -1, 2, 4)$ using the definition.
2. Compute $\mathrm{DFT}_4(-1, 3, 4, 10)$ using the definition.
3. Compute $\mathrm{DFT}_4^{-1}(0, 0, -4, 0)$ using the definition.
4. Compute $\mathrm{DFT}_4^{-1}(2, 1 - i, 0, 1 + i)$ using the definition.
5. Use the FFT algorithm to compute $\mathrm{DFT}_4(1, -1, 2, 4)$.
6. Use the FFT algorithm to compute $\mathrm{DFT}_4(-1, 3, 4, 10)$.
7. Compute $\mathrm{DFT}_4^{-1}(0, 0, -4, 0)$ using the definition.
8. Compute $\mathrm{DFT}_4^{-1}(2, 1 - i, 0, 1 + i)$ using the definition.
9. Use the IFFT algorithm to compute $\mathrm{DFT}_4^{-1}(0, 0, -4, 0)$.
10. Use the IFFT algorithm to compute $\mathrm{DFT}_4^{-1}(2, 1 - i, 0, 1 + i)$.

Solutions to FFT Core Exercises

1. $\text{DFT}_4(1,-1,2,4) = (6,-1-5i,0,-1+5i)$

2. $\text{DFT}_4(-1,3,4,10) = (16,-5-7i,-10,-5+7i)$

3. $\text{DFT}_4^{-1}(0,0,-4,0) = (-1,1,-1,1)$

4. $\text{DFT}_4^{-1}(2,1-i,0,1+i) = (1,0,0,1)$

5. $p_0(x) = 1+2x$, $\text{DFT}_2(1+2x) = (3,-1)$. Thus,
$$Y_0 = (3,-1,3,-1).$$
Also, $p_1(x) = -1+4x$, and $\text{DFT}_2(-1+4x) = (3,-5)$. Thus,
$$Y_1 = (3,-5,3,-5).$$
Furthermore, $Y_{1j} \leftarrow \omega_4^j Y_{1j}$ gives
$$Y_1 = (3,-5i,-3,5i).$$
Finally, $\text{DFT}_4(1,-1,2,4) = Y_0 + Y_1 = (6,-1-5i,0,-1+5i)$.

6. $p_0(x) = -1+4x$, $\text{DFT}_2(-1+4x) = (3,-5)$. Thus,
$$Y_0 = (3,-5,3,-5).$$
Also, $p_1(x) = 3+10x$, and $\text{DFT}_2(3+10x) = (13,-7)$. Thus,
$$Y_1 = (13,-7,13,-7).$$
Furthermore, $Y_{1j} \leftarrow \omega_4^j Y_{1j}$ gives
$$Y_1 = (13,-7i,-13,7i).$$
Finally, $\text{DFT}_4(-1,3,4,10) = Y_0 + Y_1 = (16,-5-7i,-10,-5+7i)$.

7. Input $(0,0,-4,0)$ corresponds with polynomial $p(x) = -4x^2$. Moreover,
$$p(\omega_4^{(-1)(0)}) = p(1) = -4,$$
$$p(\omega_4^{-1}) = p(-i) = 4,$$
$$p(\omega_4^{-2}) = p(-1) = -4,$$
and
$$p(\omega_4^{-3}) = p(i) = 4.$$
Thus,
$$\text{DFT}_4^{-1}(0,0,-4,0) = \frac{1}{4}(-4,4,-4,4) = (-1,1,-1,1),$$
and so $\text{DFT}_4^{-1}(0,0,-4,0) = (-1,1,-1,1)$, which corresponds with polynomial $-1+x-x^2+x^3$.

8. Input $(2, 1-i, 0, 1+i)$ corresponds with polynomial $p(x) = 2 + (1-i)x + (1+i)x^3$. Moreover,

$$p(\omega_4^{(-1)(0)}) = p(1) = 4,$$
$$p(\omega_4^{-1}) = p(-i) = 0,$$
$$p(\omega_4^{-2}) = p(-1) = 0,$$

and

$$p(\omega_4^{-3}) = p(i) = 4.$$

Thus, $\text{DFT}_4^{-1}(2, 1-i, 0, 1+i) = (1, 0, 0, 1),$, which corresponds with polynomial $1 + x^3$.

9. $p_0(x) = -4x$, $\text{DFT}_2^{-1}(-4x) = \frac{1}{2}(-4, 4) = (-2, 2)$. Thus,

$$C_0 = (-2, 2, -2, 2).$$

Also, $p_1(x) = 0$, and $\text{DFT}_2^{-1}(0) = (0, 0)$. Thus,

$$C_1 = (0, 0, 0, 0).$$

Furthermore, $C_{1j} \leftarrow \omega_4^{-j} C_{1j}$ gives

$$C_1 = (0, 0, 0, 0).$$

Finally, $\text{DFT}_4^{-1}(0, 0, -4, 0) = \frac{1}{2}(C_0 + C_1) = \frac{1}{2}(-2, 2, -2, 2) = (-1, 1, -1, 1)$, which corresponds with polynomial $-1 + x - x^2 + x^3$.

10. $p_0(x) = 2$, $\text{DFT}_2^{-1}(2) = \frac{1}{2}(2, 2) = (1, 1)$. Thus,

$$C_0 = (1, 1, 1, 1).$$

Also, $p_1(x) = (1-i) + (1+i)x$, and $\text{DFT}_2^{-1}((1-i) + (1+i)x) = \frac{1}{2}(2, -2i) = (1, -i)$. Thus,

$$C_1 = (1, -i, 1, -i).$$

Furthermore, $C_{1j} \leftarrow \omega_4^{-j} C_{1j}$ gives

$$C_1 = (1, -1, -1, 1).$$

Finally, $\text{DFT}_4^{-1}(2, 1-i, 0, 1+i) = \frac{1}{2}(C_0 + C_1) = (1, 0, 0, 1)$, which corresponds with polynomial $1 + x^3$.

Additional Exercises

A. Prove that for any two complex numbers c and d, $\overline{cd} = \bar{c}\bar{d}$

B. Write the standard form for each of the complex cube roots of unity.

C. Write the standard form for each of the 6th roots of unity. Use the standard forms to verify both parts of Proposition 3.7.

D. For the 6th roots of unity, determine the multiplicative inverse of each root, and verify that $(a + bi)(a + bi)^{-1} = 1$ through direct multiplication of the corresponding standard forms.

E. Write the standard form for each of the 8th roots of unity. Use the standard forms to verify both parts of Proposition 3.7.

F. Let $n \geq 1$, $d > 0$, and k be integers. Prove that $\omega_{dn}^{dk} = \omega_n^k$. This is called the **cancellation rule**.

G. Let n be an even positive integer. Prove that the square of each of the nth roots of unity yields the $n/2$ roots of unity. Moreover, each $n/2$ root of unity is associated with two different squares of nth roots of unity.

H. Show that $\omega_n^{n/2} = -1$, for all even $n \geq 2$.

I. For positive integer n and for integer j not divisible by n, prove that

$$\sum_{k=0}^{n-1} \omega_n^{jk} = 0.$$

Hint: use the geometric series formula

$$\sum_{k=0}^{n-1} a^k = \frac{a^n - 1}{a - 1},$$

which is valid when a is a complex number.

J. Show the sequence of polynomials that are evaluated when evaluating $p(x) = x^3 - 3x^2 + 5x - 6$ using Horner's algorithm. Use the algorithm to evaluate $p(-2)$.

K. Show the sequence of polynomials that are evaluated when evaluating $p(x) = 2x^4 - x^3 + 2x^2 + 3x - 5$ using Horner's algorithm. Use the algorithm to evaluate $p(5)$.

L. Find the equation of the quadratic polynomial whose graph passes through the points $(2, 13)$, $(-1, 10)$, and $(3, 26)$.

M. Find the equation of the cubic polynomial whose graph passes through the points $(0, -1)$, $(1, 0)$, $(-1, -4)$, and $(2, 5)$.

Solutions to Additional Exercises

A. Let $c = a + bi$, and $d = e + fi$. Then

$$\overline{cd} = \overline{(ae - bf) + i(af + be)} = (ae - bf) - i(af + be).$$

On the other hand,

$$\overline{c}\overline{d} = (a - bi)(e - fi) = (ae - bf) + i(-af - be) = (ae - bf) - i(af + be),$$

which proves the claim.

B. For $j = 0$,

$$e^{\frac{(2\pi)(0)i}{3}} = 1.$$

For $j = 1$,

$$e^{\frac{2\pi i}{3}} = -1/2 + \frac{\sqrt{3}i}{2}.$$

For $j = 2$,

$$e^{\frac{4\pi i}{3}} = -1/2 - \frac{\sqrt{3}i}{2}.$$

C. For $j = 0$,

$$e^{\frac{(2\pi)(0)i}{6}} = 1.$$

For $j = 1$,

$$e^{\frac{2\pi i}{6}} = \frac{1}{2} + \frac{\sqrt{3}i}{2}.$$

For $j = 2$,

$$e^{\frac{4\pi i}{6}} = \frac{-1}{2} + \frac{\sqrt{3}i}{2}.$$

For $j = 3$,

$$e^{\frac{6\pi i}{6}} = e^{\pi i} = -1.$$

For $j = 4$,

$$e^{\frac{8\pi i}{6}} = \frac{-1}{2} - \frac{\sqrt{3}i}{2}.$$

For $j = 5$,

$$e^{\frac{10\pi i}{6}} = \frac{1}{2} - \frac{\sqrt{3}i}{2}.$$

Notice that, for $j = 0, 1, 2$, $\omega_6^j = -\omega_6^{3+j}$. For example,

$$\omega_6^1 = \frac{1}{2} + \frac{\sqrt{3}i}{2} = -(-\frac{1}{2} - \frac{\sqrt{3}i}{2}) = \omega_6^4.$$

Finally, computing the squares of each sixth root of unity yields the numbers 1, $\frac{-1}{2} + \frac{\sqrt{3}i}{2}$, $\frac{-1}{2} - \frac{\sqrt{3}i}{2}$ which are the third roots of unity.

D. We have

$$\omega_6^0 \cdot \omega_6^0 = (1)(1) = 1.$$

$$\omega_6^1 \cdot \omega_6^5 = (\frac{1}{2} + \frac{\sqrt{3}i}{2})(\frac{1}{2} - \frac{\sqrt{3}i}{2} = 1,$$

$$\omega_6^2 \cdot \omega_6^4 = (-\frac{1}{2} + \frac{\sqrt{3}i}{2})(-\frac{1}{2} - \frac{\sqrt{3}i}{2} = 1,$$

and

$$\omega_6^3 \cdot \omega_6^3 = (-1)(-1) = 1.$$

E. For $j = 0$,

$$e^{\frac{(2\pi)(0)i}{3}} = 1.$$

For $j = 1$,

$$e^{\frac{\pi i}{4}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}i}{2}.$$

For $j = 2$,

$$e^{\frac{\pi i}{2}} = i.$$

For $j = 3$,

$$e^{\frac{3\pi i}{4}} = \frac{-\sqrt{2}}{2} + \frac{\sqrt{2}i}{2}.$$

For $j = 4$,

$$e^{\pi i} = -1.$$

For $j = 5$,

$$e^{\frac{5\pi i}{4}} = \frac{-\sqrt{2}}{2} + \frac{-\sqrt{2}i}{2}.$$

For $j = 6$,

$$e^{\frac{3\pi i}{2}} = -i.$$

For $j = 7$,

$$e^{\frac{7\pi i}{4}} = \frac{\sqrt{2}}{2} + \frac{-\sqrt{2}i}{2}.$$

Notice that, for $j = 0, 1, 2, 3$, $\omega_8^j = -\omega_8^{4+j}$. For example,

$$\omega_8^2 = i = -(-i) = \omega_8^6.$$

Finally, computing the squares of each eighth root of unity yields the numbers 1, i, -1, and $-i$ which are exactly the fourth roots of unity.

F. By definition,

$$\omega_{dn}^{dk} = e^{\frac{2\pi idk}{dn}} = e^{\frac{2\pi ik}{n}} = \omega_n^k.$$

G. For $j = 0, \dots, n-1$,

$$(\omega_n^j)^2 = \omega_n^j \omega_n^j = \omega_n^{2j} = \omega_{n/2}^j,$$

where the last equality is due to the cancellation rule from Exercise 6. Thus the square of an nth root of unity is indeed an $n/2$ root of unity. Moreover, notice that j ranges from 0 to $n-1$. By definition, when j ranges from 0 to $n/2-1$, we obtain each $n/2$ root of unity. Then, due to the cyclic nature of the roots unity, when j ranges from $n/2$ to $n-1$, we once again obtain each $n/2$ root of unity. Therefore, each $n/2$ root of unity $\omega_{n/2}^j$ is the square of exactly two different nth-roots of unity, namely $(\omega_{n/2}^j)^2$ and $(\omega_{n/2}^{j+n/2})^2$.

H. We have, for even $n \geq 2$,

$$\omega_n^{n/2} = e^{(2\pi i/n)n/2} = e^{\pi i} = \cos\pi + i\sin\pi = -1.$$

I. Using the geometric series formula

$$\sum_{k=0}^{n-1} a^k = \frac{a^n - 1}{a - 1},$$

we have

$$\sum_{k=0}^{n-1} (\omega_n^j)^k = \sum_{k=0}^{n-1} \omega_n^{jk} =$$

$$\frac{\omega_n^{jn} - 1}{\omega_n^j - 1} = \frac{\omega_1^j - 1}{\omega_n^j - 1} = \frac{1 - 1}{\omega_n^j - 1} = 0,$$

where the first equality is due to the cancellation rule, and the 2nd to last equality is due to the fact that $\omega_1^1 = 1$. Notice also that the denominator is not equal to zero, since we assumed j is not divisible by n; i.e. $j \not\equiv 0 \bmod n$.

J. $p_0(x) = 1$, $p_1(x) = xp_0(x) - 3 = x - 3$, $p_2(x) = xp_1(x) + 5 = x^2 - 3x + 5$, $p_3(x) = xp_2(x) - 6 = x^3 - 3x^2 + 5x - 6$. $p_0(-2) = 1$, $p_1(-2) = -2(1) - 3 = -5$, $p_2(-2) = -2(-5) + 5 = 15$, $p_3(-2) = -2(15) - 6 = -36$.

K. $p_0(x) = 2$, $p_1(x) = xp_0(x) - 1 = 2x - 1$, $p_2(x) = xp_1(x) + 2 = 2x^2 - x + 2$, $p_3(x) = xp_2(x) + 3 = 2x^3 - x^2 + 2x + 3$, $p_4(x) = xp_3(x) - 5 = 2x^4 - x^3 + 2x^2 + 3x - 5$. $p_0(5) = 2$, $p_1(5) = 5(2) - 1 = 9$, $p_2(5) = 5(9) + 2 = 47$, $p_3(5) = 5(47) + 3 = 238$, $p_4(5) = 5(238) - 5 = 1185$.

L. We desire a polynomial of the form $c_0 + c_1x + c_2x^2$. The three points imply the following system of equations.

$$c_0 + 2c_1 + 4c_2 = 13$$

$$c_0 - c_1 + c_2 = 10$$

$$c_0 + 3c_1 + 9c_2 = 26$$

Solving this system gives the polynomial $5 - 2x + 3x^2$.

M. We desire a polynomial of the form $c_0 + c_1x + c_2x^2 + c_3x^3$. The four points imply the following system of equations.

$$c_0 = -1$$

$$c_0 + c_1 + c_2 + c_3 = 0$$

$$c_0 - c_1 + c_2 - c_3 = -4$$

$$c_0 + 2c_1 + 4c_2 + 8c_3 = 5$$

Solving this system gives the polynomial $-1 + x - x^2 + x^3$.